OPTIMAL CONTROL USING KRONECKER-PRODUCT LYAPUNOV-FUNCTION BASED TECHNIQUE: APPLICATION TO A 2-DOF HELICOPTER MODEL SET-UP

COMMANDE OPTIMALE AVEC TECHNIQUE BASEÉ SUR FONCTION DE LYAPUNOV ET PRODUIT DE KRONECKER: APPLICATION À UN DISPOSITIF DE MODÈLE D'HÉLICOPTÈRE À 2-DDL

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By

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To the soul of my father who just passed away…

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To all my family and friends, thank you for your encouragement.

Abstract

In this work, we present the mathematical framework of the Kronecker product (KP) algebra and the optimal control theory. Using the advantage of such mathematical properties we present the optimal control of polynomial systems. We start by the algorithm of calculus of the optimal control law and we illustrate its efficiency through the application to some nonlinear plants. Also, we develop a new method called Lyapunov-function-based optimal control using KP presenting the advantage of guaranteeing the stability of the closed loop system by solving a linear matrix inequality (LMI) feasibility problem. We present the algorithm of calculus of such stabilizing control law and we illustrate its efficiency through nonlinear plants. The experimental part of this work was conducted on a two-degree-of-freedom (2-DOF) helicopter-based model set-up, in which we run many experiments for different desired trajectories to test the efficiency of the proposed method.

Résumé

Dans ce travail, nous présentons la base mathématique de l'algèbre du produit de Kronecker et la théorie de la commande optimale. En profitant de l'avantage de ses propriétés mathématiques, nous présentons la commande optimale des systèmes polynomiaux en utilisant le produit Kronecker. Nous commençons par l'algorithme de calcul de la loi de commande et nous illustrons son efficacité à travers son application à des dynamiques non linéaires. De plus, nous développons une nouvelle méthode, appelée commande optimale basée la fonction de Lyapunov en utilisant le produit Kronecker, qui présente l'avantage de garantir la stabilité du système en résolvant un problème de faisabilité d'inégalité matricielle linéaire. Nous présentons l'algorithme de calcul de la loi de commande optimale stabilisante et nous illustrons son efficacité à travers son application à deux dynamiques non linéaires. La partie expérimentale de ce travail est conduite sur un système simplifié d'hélicoptère a deux degrés de liberté. Nous avons expérimenté et observé la réponse du système pour différentes trajectoires pour tester l'efficacité de la méthode proposée.

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List of Symbols

x^T	: Transpose of x
$\dot{x}$	: First derivative of x *w.r.t.* time
$\ddot{x}$	: Second derivative of x *w.r.t.* time
¢ x	: Small variation of x
$\frac{df}{dx}$	: Derivative of f *w.r.t.* x
$\frac{\partial f}{\partial x}$	: Partial derivative f *w.r.t.* x
$Df(x)^T$	: Gradient of the real valuated function $f(x)$
$D^2 f(x)$	: Hessian of the real valuated function $f(x)$
$\int_{x_1}^{x_2} f(x)dx$	: Integral of x between x_1 and x_2
$\otimes$	: Kronecker product
$E_{ik}^{(p\times q)}$	: Elementary matrix
$U_{p\times q}$	: Permutation matrix
$vec(.)$	: Vec operator (matrix to vector)
$mat(.)$	: Mat operator (vector to matrix)
$x^{\lvert i \rvert}$	: i^{th} power of x in terms of KP

$\tilde{x}^{|i|}$: Non redundant i^{th} power of x in terms of KP

T_j : Transformation matrix

T_j^+ : Moore-Penrose pseudo inverse of T_j

I_n : Identity matrix of order n

$D_j^{(n)}$: The square j-differential Kronecker matrix of $\mathbb{R}^{n^j \times n^j}$

List of Acronyms

KP	: Kronecker product
VPS	: Vector power series
LF	: Lyapunov function
DOF	: Degree of freedom
ARE	: Algebraic Ricatti equation
LQR	: Linear quadratic regulator
LTV	: Linear time varying
LTI	: Linear time invaryi ng
HJE	: Hamilton Jacobi equation
HJB	: Hamilton Jacobi Bellman equation
SDR	: State dependent Ricatti equation

1 Introduction

Every dynamic system can be described by a set of outputs which are functions of a set of inputs. These functions are called the dynamics of the system (having the general form of $\dot{x} = f\left(x,u\right)$, where f is a nonlinear vector function). These dynamics are in most cases complex, highly nonlinear and hard to solve. That's why in the beginning of control engineering field, researchers approximate the dynamics to a linear form and design linear controllers, among which some popular techniques are still used until today, for example the Proportional-Integral-Derivative (PID) one. The linear control strategy has some limitations since it is not taking into account the best approximation to represent the real dynamics of the system, it has a reduced domain of attraction and it does not guarantee the stability of the closed loop system in general. To overcome those issues, mathematicians and control engineers developed various nonlinear control strategies. As a special case, they developed the optimal control for a large class of nonlinear systems (which can be written in the form of $\dot{x} = f\left(x\right)x + g\left(x\right)u$, where f and g are nonlinear functions). The aim of this optimal control strategy is to find the control law based on the minimization of a certain performance index. Since most of the nonlinear dynamics cannot be written in the above form, researchers use the advantage of the KP algebra and the vector power tensor [1-2] to approximate nonlinear dynamics by polynomials (in the form of $\dot{x} = \sum_{i=1}^{n} f_i x^{|i|}$) using the Taylor series development. This method is called optimal control of polynomial systems using KP [3]. As it will be shown, this method has limitations too since it does not guarantee the stability of the closed loop system. In fact, the choice of the cost function approximation does not satisfy the conditions of the Lyapunov stability [4]. This leads to the main contribution of this work which is to present a new method called optimal control of polynomial systems using KP-based Lyapunov functions (LF). This method is based in the fact of choosing the cost function to be minimised in a quadratic form and depending on an extra real scalar to satisfy the conditions of Lyapunov stability and to guarantee the asymptotic stability of the closed loop system.

To present the design process and illustrate the efficiency of this new method, a general theoretical framework and practical application should be presented. This thesis will be organised as follows: First, after introducing the thesis topic and its main objectives. We will present in chapter 2 the state of the art of the main topics of this work, *i.e.*, optimal control theory, nonlinear control, stability analysis of polynomial systems, the KP algebra and its applications. In chapter 3, first we will present the KP algebra. We will recall its basic definitions, proprieties and the proof of new results (theorems and lemmas). Then, we will introduce the vector power series (VPS) motivation to present the best approximation of nonlinear functions, and we will illustrate through many examples. In chapter 4, we will present the optimal control theory framework. We will begin by the presentation of the optimization problem without constraints then with equality constraints, using the Lagrange multipliers method. The optimal control problem will be transformed into solving the so-called Hamilton Jacobi equation [5]. Then, we will show how the resolution of this equation leads to determine the gain matrix of a popular controller called Linear Quadratic Regulator (LQR). In chapter 5, we will study the optimal control problem of a special class of nonlinear systems which can be written in a polynomial form in terms of KP tensor. We will state the problem. Then, we will present the equation of approximation and the algorithms to determine the gain matrices of the optimal control law. At the end of this chapter, we will illustrate the efficiency of this method through its application to different nonlinear plants (scalar examples, F8 fighter and Maglev set-up models). We will show the simulation results and performance improvements obtained by the KP controllers versus the linear controllers. Since the KP-based controllers do not guarantee the stability of the closed loop system, we have the idea to design a stabilizing method by choosing the cost function to be minimized in a quadratic form and depending on a real scalar to satisfy the conditions of a Lyapunov candidate function. This topic will be detailed in chapter 6. In fact, we will introduce the statement of the problem then the equation of approximation in which the initial problem is transformed into a set of decoupled linear equations. Then, we will present the algorithm of calculus of the gain matrices of the state feedback stabilizing control. Next, we will discuss the stability of the closed loop system. We will show how the stability problem will be transformed into solving an LMI problem. Finally, to illustrate the efficiency of this new method, we present the simulation results of nonlinear plants: a scalar example and the F8 fighter model. As the real behaviour may differ from the predicted one through the simulations, we will present in chapter 7 an application to a real plant: a 2-DOF helicopter-model-based set-up from Quanser Inc. After the introduction, we will begin this chapter by presenting a description of the set up, then we present its real dynamics, the linearized and its high order polynomial approximations. Next based on those approximations, we will present the control design which will be used to run simulations and experiments for different desired trajectories. Finally, in chapter 8, we will conclude this thesis and present its main contributions and results.

Statement of contribution: Through this work, the main contributions are: the development of a new method to design a stabilizing controller for the polynomial dynamic systems, called Kronecker-product-Lyapunov-function-based technique, the statement with proofs of some theorems and lemmas related to KP algebra useful for the design process, and the implementation of the proposed technique to an actual electro-mechanical set-up. To our best knowledge the KP-based technique has been rarely tested for real-time control process.

2 State of the art

2.1 Introduction

The main framework of this thesis is related to three items. First, we deal with the optimal control theory of nonlinear systems. In particular we treat the case of polynomial systems since any nonlinear dynamics can be written in multivariable power series form within a certain degree of approximation error. Then, we carry out the advantage of the KP algebra to write the dynamics in a compact form and make some appropriate mathematical manipulations to design a nonlinear controller. As any research topic, we presented in this chapter the state of the art related to those topics. In section 2.2, we will present an overview of the optimal control history. Then, in section 2.3, we will introduce the framework of the optimal control theory. In section 2.4, we will present the state of the art for the nonlinear control and stability analysis of polynomial systems. In section 2.5, we will present the Kronecker product algebra and its applications. Finally, in section 2.6, we will conclude this chapter.

2.2 Optimal control history

Sargent (2000) stated in one of the most exhaustive research papers on optimal control that: "Optimal control theory is an outcome of the calculus of variations, with a history stretching back over three hundred and sixty years, but interest in it really mushroomed only with the advent of computer, launched by the spectacular successes of optimal trajectory prediction in aerospace applications in the early 1960s" [17].

According to Sargent (2000) in [17], the optimal control birth was in 1638, when Galileo posted the two shape problems: the catenary and the brachistochrone. The catenary system is a heavy chain suspended between two points, and the brachistochrone system is a wire such that a bead sliding along it under gravity. But despite of his efforts and conjectures, the solutions of those two problems were incorrect. In 1685, Newton presented a solution to the nose shape of a projectile providing minimum drag problem and published the results in 1694. In 1696, five mathematicians (Newton, Bernoulli, Leibnitz, De l'Hopital and Tschirnhaus) solved the Brachistochrone problem and Bernoulli published the solution in 1697.

This publication has risen the interest in the mathematics community to solve this type of problems. This interest has yielded a number of ideas and results for such problems. In 1744, Euler, a student of Bernoulli collected those ideas in a book. Based on the observation that "nothing at all takes place in the universe in which some rules of maximum or minimum does not appear", Euler formulated in 1744, the problem in general terms as one of finding the curve $x(t)$ over the interval $[a,b]$ with given values $x(a)$, $x(b)$, which minimizes $J=\int_a^b L(t,x(t),\dot{x}(t))dt$ for some given function $L(t,x(t),\dot{x}(t))$. In 1755 and using his "undetermined multipliers", Lagrange described the first analytical approach based on perturbations or "variations" of the optimal curve. This led to the "Euler-Lagrange equation" which represents the first order necessary condition. In 1786, Legendre studied the second variation and determined the second order necessary condition of optimality for the scalar case. Clebsch extended later this condition to the vector case which leads to the Legendre-Clebsch condition requiring the same matrix to be nonnegative definite along the optimal trajectory. Later, Hamilton introduced the "Hamiltonian function", transformed the problem to a variational principle, and he expressed the latter through a pair of partial differential equations. And, in 1838, Jacobi showed that it could be written in a more compact form known as Hamilton-Jacobi equation (HJE). Then, Weirestrass introduced the "excess function", in which he considered the special case where $L(t,x(t),\dot{x}(t))$ is positive homogenous. Later, Caratheodory showed that his excess function is positive if and only if the second derivative of the Hamiltonian function is positive and by this he confirmed the sufficiency of the Hamilton-Jacobi solution even under strong variation. Based on Caratheodory work to establish the existence of optimal trajectories, Tonelli treated the problem of existence and he showed that this existence is guaranteed if the function $L(t,x(t),\dot{x}(t))$ is convex. Then, by the restriction of the class of admissible functions $\dot{x}(\cdot)$ satisfying the set of equations $g(t,x(t),\dot{x}(t))=0$ known as general set of differential algebraic equations and the sufficient condition were imposed to ensure that there exists functions $\dot{x}=f(t,x(t))$ satisfying $g(t,x(t),\dot{x}(t))=0$. The resulting problem was named problem of Lagrange, in which a solution presented by the introduction of Lagrange multipliers, and by considering constraints of the form $\dot{x}=f(t,x(t),u(t))$, where the parameters $u(t)$ or "controls" can be chosen at each instant $t\in[a,b]$. This yields the "optimal control problem" as follows: Find $u(\cdot)$ on (a,b) to minimize $J=\int_a^b L(t,x(t),\dot{x}(t))dt$ subject to $\dot{x}=f(t,x(t),u(t))$,

$t \in (a,b)$, $x(a)$ and $x(b)$ are given. In 1950, and based on the early work of Hamilton and Jacobi in which they established the HJE, Bellman *et al.* developed the Hamilton-Jacobi-Bellman (HJB) equation in the whole state space. The latter represents a necessary and sufficient condition for an optimum solution [18]. Later in 1956, Pontryagin presented through his "maximum principle" a necessary condition of optimality [19].

From this point and based on the two works of Bellman and Pontryagin introduced above, many researchers and mathematicians have developed various techniques and methods to solve and to study the stability of optimal control problems for both the general and specific classes of nonlinear systems.

2.3 Optimal control theory

The optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A control problem includes a cost functional that is a function of state and control variables. An optimal control is a set of differential equations describing the paths of the control variables that minimize the cost functional. The optimal control can be derived using Pontryagin's maximum principle or by solving the HJB equation [6]. Based on those two principles, researchers have developed many control methods and strategies of optimal control theory.

In 1966, Tchamran presented in [7] an algebraic method to solve the optimal control problem for a specific class of nonlinear systems. Through a set of calculus of variations he showed how the optimal control problem leads to solving Bellman's functional equations. Then, through two illustrative examples, he showed how to solve this equation manually and give "the exact" solution of the problem. Despite his effort, this method presents a limitation since it does not guarantee the computation of the "exact" solution as well as the stability of the closed loop system. Later in 1968, Kyong and Gyftopouls proposed in [8] another direct algebraic method to solve the optimal control problem for another specific class of nonlinear systems. They transform the nonlinear equation of the optimal control problem into a set of algebraic equations through the benefit of the expansions of the Kernels of the system. Despite its advantage to solve directly the set of equations, this method does not guarantee always a solution, also the stability of the closed loop system is not studied. In a more recent work published in 1993, Goh presented a new approach to solve the nonlinear optimal control problem using a numerical approximation method based on the neural network algebra [5]. He transformed the problem into finding the optimal set of weights used to approximate the control law by using a nonlinear regression procedure. He also showed that the asymptotical stability of the closed loop system is guaranteed within a given domain of attraction by the appropriate choice of synaptic weight parameters and a given weighting matrix. Finally, he illustrated his proposed design through two nonlinear examples: the flight control system of an F8 fighter model and a two-state system example. Two years later, in 1995, Dakev *et al.* proposed in [9] a general approach to solve optimal control problems using a constrained optimization technique. This technique is based on the approximation of the input control using a piece-wise approximation vector function. They illustrated their proposed techniques through four examples: an academic example with state constraints, two-link articulated manipulator optimal trajectory problem and optimal following path problems and lifting re-entry space vehicle. Inspite of the obvious numerical solution proposed to solve the optimal control problem, the asymptotic stability of the system is not discussed on this work. Later, in 1996, Rehbock *et al.* presented in [10] the design of suboptimal controller for a specific class of nonlinear systems in the form of $\dot{x} = A(x)x + B(x)u$, where $A(x)$ and $B(x)$ are state dependent. They also showed that the closed loop system is asymptotically stable under specific conditions. They illustrate their design through two examples: an academic example with 2-DOF model and the F8 fighter flight control system. In the second part of this work, the authors studied the optimal control problem of the same class of nonlinear system, with the input control subject to bounded noise. They showed that despite of the noise signal, the system response is still bounded. To illustrate this result they presented the responses of the same two studied systems, but with a bounded noise signal in the input. One year later, in 1997, Langson and Alleyne presented in [11] an extended work for the optimal control problem for a more general class of nonlinear systems in the form of $\dot{x} = f(x) + g(x)u$. They showed through appropriate transformations and conditions on the weighting matrices of the functional cost. They illustrated their proposed design using a 3-DOF example in which they showed the robustness of the closed loop system via proper simulations. Then, they showed through an experimental setup the implementation of their method despite of the numerical issues due to the extensive calculus of the gain feedback matrix at each point along the solution trajectory. In 1999, Primbs *et al.* presented in [12] two approaches to solve the optimal control problem. In the first one, called Control Lyapunov Function (CLF), they showed that starting from the JBE and using Sontag's formula lead to the design of an optimal controller. They showed through an illustrative example the poor performance of such design and the fact it does not guarantee the stability of the system. The second approach, called Receding Horizon Control (RHC), is based on the decomposition of the time interval into finite intervals $[t, t+T]$, and the optimal control problem is solved for each of those time intervals. This resolution is required online (*i.e.*, in real time) which may cause computational issues. They showed also the lack of efficiency of this method through the same illustrative example since it shows poor performance and does not guarantee the stability of the system. More recently in 2005, Ekman treated in [13] a suboptimal control approach for bilinear systems. He presented a control law based on an approximate solution of the HJB equation. This approximation is based on the development of the partial derivative of Lyapunov equation with respect to the state vector x in terms of power series of x. He showed also that the stability of the closed loop system is not guaranteed and depends on the order of truncation of the Taylor series expansion. As an illustrative example to show the performance of the developed controller, the author presented the activated sludge process and he run the simulation for two controllers: the proposed one (Taylor series approximation) and the classic LQR. The simulation results showed a better performance for the Taylor-series-based controller in terms of settling time and overshoot. In 2008, Rafikov *et al.* presented in [14] a linear optimal control law for a class of nonlinear systems in the form of $\dot{x} = A(t)x + G(x)x + Bu$. They showed that this controller guarantee the local asymptotical stability or the global asymptotical stability for specific conditions. They illustrate their proposed method through two examples: the Duffing oscillator for which they presented through simulations the behaviour of the closed loop system and the automotive active suspension system for which they showed the performance improvement in terms of sprung and unsparing mass displacement. In 2009, Basin and Alvarez developed in [15] a sliding mode controller for a class of nonlinear systems in the form of $\dot{x} = f(x,t) + B(t)u(t)$, where the nonlinear function $f(x,t)$ can be written in a time-variant-polynomial form in the n variable state vector x. The developed control law is obtained through the minimization of the quadratic Bolza-Meyer function. They also showed that the conventional polynomial quadratic regulator fails to provide a feasible solution, whereas the sliding mode one give an optimal solution. They illustrate the advantage of their design through the simulation of a nonlinear plant and observe the behaviour of both controllers (sliding mode and polynomial quadratic regulator). The sliding mode design shows an advantage in terms of performance improvement. In 2010, Jajami *et al.* presented in [16] a new method to solve the optimal control problem for a class of nonlinear systems in the form of $\dot{x} = Ax + Bu + f(x)$. First, they transformed the optimal control problem into a nonlinear two-point-boundary-value problem (TPBVP) via Pontryagin's maximum principle. This problem is then transformed into a sequence of linear time-invariant (LTI) TPBVP (LTITPBVP) by using the homotopy method. The resolution of the LTITPBVP recursively leads to the optimal control law. Inspite of the simplicity of the algorithm and the low computation load of the proposed method, this technique has limitations in practice since it does not guarantee the stability of the system.

2.4 Nonlinear control and stability analysis of polynomial systems

The study of stability is a significant phase in the analysis and the synthesis of dynamic systems. This explains the abundance of works and publications devoted to this question since 1892, date on which Lyapunov made appear the first results of the theory of motion stability [24]. The polynomial technique of studying the stability of nonlinear systems is one of the most important developed approaches. It is based on the modeling of the considered nonlinear analytical system by a polynomial system. Notice that the class of polynomial systems is large enough to include the description of nonlinear systems and it can be simplified using the Kproduct tensor and power of vectors and matrices [25].

In 1996, Braiek presented in [20] a sufficient condition to verify the global asymptotical stability of nonlinear polynomial systems by checking the positiveness, definiteness and symmetry of a given matrix depending on transformation matrices, relating the redundant to the non-redundant power vectors, and the coefficient matrices, in which the dynamics of the system is determined as a polynomial form in terms of KP decomposition, then the system is globally asymptotically stable. Despite the usefulness of the proposed method was useful to prove the global stability of non-linear polynomial systems and the synthesis of nonlinear controllers, this technique shows only sufficient conditions. More recently, based on this framework and using the sufficient condition to check the global asymptotic stability of nonlinear polynomial systems, Ayadi and Braiek proposed in 2004 in [21] a stabilizing control law. The existence of a solution comes from a result presented in [20] transforming the sufficient condition to an LMI feasibility problem. The resolution of this LMI leads to the calculus of the gain matrices. The authors illustrated their method with an example showing the performance of such design in terms of stability. Then, in 2006, Bouzaouache and Braiek treated in [22] the global exponential stability of a class of singularly perturbed nonlinear systems composed of two subsystems (slow and fast components). First, they wrote the nonlinear terms in a polynomial form using the KP. Using the advantage of the KP algebra to write one "unified" state which is composed from slow and fast subsystem states. They made coordinate transformations to re-write the system dynamics in terms of one global vector. Then, they showed the GAS property based on an LMI problem statement. The authors studied in 2007, in [23], the stability of more general nonlinear systems. First, they assumed that any nonlinear system can be written in a polynomial form in terms of KP and VPS state vectors. Then, based on this assumption and using some KP properties, they transform the stability in the sense of Lyapunov problem to an LMI feasibility problem. They showed that any polynomial system in the form of $\dot{x}=\sum_{i=1}^{r}F_{i}x^{|i|}$ is stable if it exists a feasible solution to the LMI problem $P=P^{T}>0$ and $2^{T}\left(PM+M^{T}P\right)2<0$, where 2 and M are two appropriate matrices calculated from the dynamics of the system. In 2007, Bouzaouache *et al.* studied in [24] the stability of a class of hybrid nonlinear systems in which the dynamics is written in terms of two-state vectors: continuous and discrete ones. First, they proposed to write the hybrid system dynamics in the form of KP-based finite vector power series in terms of the continuous state vector, then thanks to the KP algebra they transformed the problem of stability in the sense of Lyapunov to an LMI feasibility problem. Two years later, in 2009, Mtar studied in [25] the global stability analysis in the sense of Lyapunov of the polynomial systems which can be written in terms of KP state vector power series with an odd order of truncation. First, they made some the KP calculations to transform the stability in the sense of Lyapunov problem to a bilinear matrix inequality (BMI) feasibility problem. Then, using the separation Lemma, the generalized SCHUR's complement and some algebraic manipulations they transform the BMI problem into an LMI feasibility one. The latter presents an advantage since it has additional degrees of freedom in terms of decision variables. In the same year, Belhouane *et al.* studied in [26] the stability of a specific class of polynomial systems written in the form $\dot{x}=\sum_{i=1}^{r}F_{i}x^{|i|}+Gu$, where G is a constant matrix and r is an odd order of truncation. This study leads to the design of a nonlinear control law that guarantee the global asymptotical stability of the system if the correspondent LMI holds. First, they write the nonlinear control law in a polynomial form in terms of the state vector, *i.e.* $u=\sum_{i=1}^{r}K_{i}x^{|i|}$, which allows through proper KP algebra manipulations the transformation of the stability in the sense of Lyapunov problem to a BMI problem. Then, by using the separation Lemma and the SCHUR's complement, they transform the BMI to an LMI feasibility problem. The resolution of the latter leads to the calculus of the gain matrices and hence the stabilizing polynomial control law. Finally, they illustrate the efficiency of their proposed design through an illustrative example. In 2009, Jemai *et al.* proposed in [27] a new method to determine the feedback nonlinear control law for the infinite horizon of the class of polynomial systems written in the form $\dot{x}=\sum_{i\geq 1}F_{i}x^{|i|}+Gu$, with G a constant matrix. This method is based on the coordinate transformation of the state and control vectors to re-write the original system in a linear form. Then, by choosing the poles of the latter, they determine the polynomial control law of the original system. Despite the advantage of easy implementation of the proposed method, it is missing the theoretical framework that shows the existence of the transformation and there is no stability study of the proposed control law. The same method of design of the same class of polynomial system was presented in 2010 by Derbali *et al.* in [28]. This design method was called fault tolerant control design. To illustrate the effectiveness of their proposed design, they applied this method to a series DC motor. The simulations showed a good performance of the controller and the stability of the system.

2.5 KP algebra and its applications

The Kronecker product was named after German mathematician Leopold Kronecker (1823-1891). It is very important in the areas of linear algebra and signal processing and it has wide applications in systems theory, matrix calculus, matrix equations, system identification, and other special fields [29].

In 1978, Brewer presented in [1] an excellent review and new algebraic proprieties of the Kproduct tensor. First, he reviewed the basic definitions of the KP, the unit vectors, the elementary matrix, the permutation matrix and the vector operator, denoted by $vec(\cdot)$, as well as some algebraic proprieties of the permutation matrices and basic algebraic proprieties in terms of associability, commutability and transpose of the KP and the Kronecker sum. Then, he presented the basic algebraic proprieties of the $vec(\cdot)$ operator also in terms of associability and commutability and transpose. In the second part of the paper, he presented the basic differentiability proprieties of the KP and its relation to the permutation matrices and $vec(\cdot)$ operator. These properties will be very useful on the differentiation of the Lyapunov candidate function for the stability study of the polynomial systems. In 2000 Van-Loan treated in [30] some applications of the KP to solve some mathematical problems. First, he presented some basic proprieties that were used to solve the Sylvester equation problem and the Lyapunov problem which is a particular case of the latter. Then, he showed that there is a solution of the least squares problem by making the appropriate decomposition and using the advantage of KP to apply the necessary algebraic transformations. Another application of the KP algebra is to solve the tensor product issues in approximations and interpolations. Also, he showed that using some KP proprieties leads to the design of fast transformation algorithms. In 2004, Laub treated in a chapter of his book [31], the KP algebra and some of its applications in matrix calculus. First, he presented some definitions, then he presented many useful proprieties, and as an illustrative application in matrix calculus, he presented how to use the KP properties to solve the Sylvester and Lyapunov equations which are widely used to solve control theory problems. In 2007, Liv and Trenkler presented in [32] a general theoretical overview of several matrix products such as Hadamard, Kronecker, Khatri-Rao, Tracy-Singh, the Khatri-Rao sum, the Tracy-

Singh sum and the vector cross, some of their proprieties and some relations between two or more matrix products as well as some of their applications. First, they presented the definition of the above cited matrix products and sums. Then, they presented the relations between the different products; in particular, between the Hadamard and Kronecker products, the Kathri-Rao and Tracy-Singh products, the Tracy-Singh and Kronecker products and the Kronecker and vector cross products. They presented some equality properties involving three or more matrices and one or more types of product. Then, they presented some inequality proprieties of the Hadamard product, KP, Khatri-Rao product, Tracy-Singh product, vector cross product and Khatri-Rao sum involving one or more matrices. Finally, they presented the application of the Khatri-Rao product to study variances in statistics and econometrics, the multi-way models and algorithms in multivariate statistics, psychometrics, engineering, food, and chemical sciences, and to design more reliable transmission antenna in signal processing. Then, they showed the use of Kronecker and Hadamard products to solve linear matrix equations and particularly the generalized Lyapunov equation which is very useful in control theory. Kaam and Nagy proposed in [33] a method called singular value decomposition (SVD) as a less expensive calculation method in image restoration. They use some KP proprieties to transform the main problem which is expensive in matrix calculus to another problem less laborious. They illustrate the advantage of their proposed method via an example of image restoration of satellite images.

2.6 Conclusion

The main objective of this chapter was to present the state of the art for the three main topics treated in this thesis, the optimal control theory, the nonlinear control of polynomial systems and the KP algebra. In section 2.1, we introduced this chapter. Then, in section 2.2, we presented the optimal control history and its roots. In section 2.3, we presented the optimal control framework state of the art. In section 2.4, we presented a literature review of the nonlinear control and stability analysis of polynomial systems. The KP algebra history and its applications was presented in section 2.5. Finally in section 2.6, we conclude this chapter.

The present chapter provides a historical background of the state of the art of the optimal control theory and the application of the KP algebra to nonlinear control techniques. Nonetheless, the application of this tensor to optimal control was very limited and rarely implemented with actual devices. Also the study of the stability of the closed loop optimal control system within such methodology was missing. In fact the main contribution of the modified KP-based framework that will be discussed in this thesis will be the possible investigation of stability. Note the latter won’t be discussed in details in this thesis, but the presented framework will make it possible. More details about this subject are presented in [57].

3 Kronecker product algebra

3.1 Introduction

When we multiply two matrices together, we generally use the conventional multiplication method. This type of matrix multiplication is commonly used in algebra and represents the composition of two linear transformations that are represented by the two matrices [34]. There is a size restriction when performing this type of matrix multiplication. The number of columns of the first matrix must be the same as the number of rows in the second matrix from the left. Also, this multiplication is not commutative in general. While this common type of matrix multiplication remains very useful, it is not unique.

The Hadamard product, denoted by the symbol $\circ$, is another type of matrix multiplication [34]. In this case, the two matrices are multiplied in the same way as with the conventional matrix addition. For this multiplication, the two matrices are required to be of the same size. The resulting matrix product is formed by multiplying the corresponding entries of the two matrices together. One useful fact about this type of matrix multiplication is that it is commutative. This product is useful in several areas of study, such us the association schemes in combinatorial theory and weak minimum principle in partial differential equations.

The KP, denoted by the symbol $\otimes$, also known as the direct product or the tensor product has an interesting advantage over the previously discussed matrix products. The dimensions of the two matrices being multiplied together do not need to have any relation to each other [34]. Many important proprieties of this product will be presented in this chapter and used in the next chapters. This kind of product is used in several areas of study such us signal and image processing, semi definite programming and quantum computing [34].

This chapter deals with the KP framework, we start by presenting its definition in section 3.2. In section 3.3, we list some proprieties of the KP as depicted from the literature [1-35]. In addition, some new proprieties will be proposed with their proofs [35]. Finally, section 3.4 is devoted to the motivation of the work that we are doing which is essentially based on the function approximation, in order to assess the reader with the point of view of this work.

3.2 Definitions

In this section, we introduce the KP with some further related notations.

Definition 3.1: Given two matrices A and B of dimensions $(p \times q)$ and $(m \times n)$ respectively, the ***Kronecker Product*** of A and B denoted by $A \otimes B$ is a $(pm \times qn)$-matrix defined by [1]:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1q}B \\ a_{21}B & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{p1}B & a_{p2}B & \cdots & a_{pq}B \end{bmatrix} \tag{3.1}$$

where a_{ik} is the $i-k$ element of A.

Definition 3.2: The ***elementary matrix*** $E_{ik}^{(p \times q)}$ of dimensions $(p \times q)$, which is "1" in the ik^{th}-element, and is zero elsewhere that is [1]:

$$E_{ik}^{(p \times q)} = e_i e_k^T = e_i^{(p)} e_k^{(q)T} \tag{3.2}$$

where $e_i^{(p)}$ is the p-dimensional column vector which is "1" in the k element and zero elsewhere and is called the unit vector.

Definition 3.3: The ***permutation matrix*** is a square $(pq \times pq)$-matrix which has precisely a single "1" in each row and each column, defined by [1]:

$$U_{p \times q} = \sum_{i=1}^{p} \sum_{k=1}^{q} E_{ik}^{(p \times q)} \otimes E_{ki}^{(q \times p)} \tag{3.3}$$

Definition 3.4: The $vec(.)$ ***operator*** corresponds to the vector valued is the operator which converts a $(p \times q)$-matrix $A = \left[A_{.1} \middle| A_{.2} \middle| \quad \cdots \quad \middle| A_{.q} \right]$ into a vector of dimensions $(pq \times 1)$, defined by [1]:

$$vec(A) = \begin{bmatrix} A_{.1} \\ A_{.2} \\ \vdots \\ A_{.q} \end{bmatrix} \tag{3.4}$$

where $A_{.k}$ is the k^{th} column of the matrix A.

Definition 3.5: Given a vector V of dimension $p = n.m$ and a matrix M of dimensions $(n \times m)$ verifying $V = vec(M)$, the $mat(.)$ ***operator*** is the function transforming the vector V into the matrix M and denoted by:

$$M = mat_{n\times m}(V) \tag{3.5}$$

This notation is proposed to simplify the representation of some block matrices that will be deduced from the new optimal control design discussed in chapter 6.

Definition 3.6: Rotella and Tunguy [3] introduce the so called the ***non-redundant*** j-power $\tilde{x}^{|j|}$ of a vector $x = \begin{bmatrix} x_1 & \dots & x_q \end{bmatrix}^T$ of $\mathbb{R}^{q\times 1}$ defined by

$$\tilde{x}^{|1|} = x^{|1|} = x \tag{3.6}$$

$\forall \; j \geq 2$,

$$\tilde{x}^{|j|} = \begin{bmatrix} x_1^j & x_1^{j-1}x_2 & \dots & x_1^{j-1}x_q & x_1^{j-2}x_2^2 x_1^{j-2}x_2x_3 & \dots & x_1^{j-2}x_2x_q & \dots & x_1^{j-2}x_q^2 & x_1^{j-3}x_2^3 & x_q^j \end{bmatrix}^T \tag{3.7}$$

The relation between the non-redundant j-power of the vector $\tilde{x}^{|j|}$ and j-power of the vector $x^{|j|}$ is defined by

$$x^{|j|} = T_j \tilde{x}^{|j|} \tag{3.8}$$

where T_j is a transformation matrix. Hence, we can write

$$\tilde{x}^{|j|} = T_j^+ x^{|j|} \tag{3.9}$$

where T_j^+ is the Moore-Penrose Pseudo-Inverse of T_j defined by [3]

$$T_j^+ = \left(T_j^T T_j\right)^{-1} T_j^T \tag{3.10}$$

Definition 3.7: Rotella and Tunguy (1988) define the binomial coefficients $\sim_p$ as $\forall j \in \mathbb{N}, \quad \exists ! T_j \in \mathbb{R}^{n^j \times \sim_j}, \quad \sim_j = \binom{n+j-1}{n}$ [3].

3.3 Properties

In this section, we present the main properties that will be used in the next chapters. Some of these proprieties are presented in [1-2] (refer to Theorems 3.1 to 3.16), while the proofs for Theorems 3.17 and 3.18 and Lemmas 3.1, 3.2 and 3.3 will be shown. Consider the following matrices of appropriate dimensions: A $(p \times q)$, B $(s \times t)$, C $(r \times l)$, D $(q \times s)$, F $(q \times u)$, G $(t \times u)$, H $(p \times q)$, M $(m \times m)$, N $(n \times n)$, R $(s \times t)$. $I_.$ is the identity matrix, $U_{.\times.}$ is the permutation matrix and x and y are given vectors of dimensions p and q, respectively. We recall the following theorems mainly discussed in [1-3]:

Theorem 3.1: We have [1]

$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \tag{3.11}$$

Theorem 3.2: We have [1]

$$(A \otimes B)^T = A^T \otimes B^T \tag{3.12}$$

Theorem 3.3: We have [1]

$$(A \otimes B)(D \otimes G) = (AD) \otimes (BG) \tag{3.13}$$

Theorem 3.4: We have [1]

$$B \otimes A = U_{s \times p} (A \otimes B) U_{q \times t} \tag{3.14}$$

Theorem 3.5: We have [1]

$$vec\left(A^T\right) = U_{p\times q} vec\left(A\right) \tag{3.15}$$

Theorem 3.6: We have [1]

$$vec\left(A+H\right) = vec\left(A\right) + vec\left(H\right) \tag{3.16}$$

Theorem 3.7: We have [1]

$$vec\left(AD\right) = \left(I_s \otimes A\right) vec\left(D\right) = \left(D^T \otimes I_p\right) vec\left(A\right) = \left(D^T \otimes A\right) vec\left(I_q\right) \tag{3.17}$$

Theorem 3.8: We have [1]

$$vec\left(ADB\right) = \left(B^T \otimes A\right) vec\left(D\right) \tag{3.18}$$

Theorem 3.9: We have [1]

$$\left(\frac{\partial A}{\partial B}\right)^T = \frac{\partial A^T}{\partial B^T} \tag{3.19}$$

Theorem 3.10: We have [1]

$$\frac{\partial\left(AF\right)}{\partial B} = \frac{\partial A}{\partial B}\left(I_t \otimes F\right) + \left(I_s \otimes A\right)\frac{\partial F}{\partial B} \tag{3.20}$$

Theorem 3.11: We have [1]

$$\frac{\partial\left(A \otimes C\right)}{\partial B} = \frac{\partial A}{\partial B} \otimes C + \left(I_s \otimes U_{p\times r}\right)\left(\frac{\partial C}{\partial B} \otimes A\right)\left(I_t \otimes U_{l\times q}\right) \tag{3.21}$$

Theorem 3.12: We have [1]

$$x^T Ay = \left[vec^T\left(A^T\right)\right]\left(x \otimes y\right) = \left[vec^T\left(A\right)\right]\left(y \otimes x\right) \tag{3.22}$$

Theorem 3.13: We have [1]

$$\frac{\partial y^T}{\partial y} = I_q \tag{3.23}$$

Theorem 3.14: We have [1]

$$U_{p\times q}^T = U_{p\times q}^{-1} = U_{q\times p} \tag{3.24}$$

Theorem 3.15: We have [1]

$$U_{p\times 1}^T = U_{1\times p} = I_p \tag{3.25}$$

Theorem 3.16: We have [1]

$$U_{st\times n} = U_{s\times tn} \cdot U_{t\times ns} = U_{t\times ns} \cdot U_{s\times tn} \tag{3.26}$$

Theorem 3.17: We have [1]

$$x^{|i|} \otimes x^{|j|} = x^{|i+j|} \tag{3.27}$$

Theorem 3.18: For any integer $n > 1$ and for any nonzero integer p, we have

$$U_{n^p\times n} = U_{n\times n^p}^{\,p} \tag{3.28}$$

Proof of Theorem 3.18: See Appendix A – section A-1

Theorem 3.19: For all integers $n > 1$, $\left(I_{n^{p+1}} + U_{n^p\times n}\right)$ is regular for p even and singular for p odd. Note that for $n = 1$, $\left(I_{n^{p+1}} + U_{n^p\times n}\right)$ is a nonzero integer.

Proof of Theorem 3.19: Theorem 3.19 is stated in [3] without proof, but this propriety can be checked numerically. In Appendix A – section A-2, we prove this result.

Lemma 3.1: $\forall\, j \in \mathbb{N}\setminus\{0\}$ and $\forall x \in \mathbb{R}^n$,

$$\frac{\partial x^{|j|}}{\partial x^T} = D_j^{(n)} \cdot \left(I_n \otimes x^{|j-1|}\right) \tag{3.29}$$

where $D_j^{(n)} \in \mathbb{R}^{n^j}$ is given by

$$D_j^{(n)} = \sum_{i=0}^{j-1} U_{n^i \times n} \otimes I_{n^{j-i-1}} \tag{3.30}$$

Note that

$$D_1^{(p)} = I_{n^p} \tag{3.31}$$

Proof of Lemma 3.1: See Appendix A – section A-3

Lemma 3.2: For all $x \in \mathbb{R}^k$, $y \in \mathbb{R}^l$ and $A \in \mathbb{R}^{nk} \times \mathbb{R}^l$, we have

$$\left(I_n \otimes x^T\right) Ay = \left(I_n \otimes vec^T\left(A^T\right)\right)\left(vec\left(I_n\right) \otimes I_{kl}\right)\left(x \otimes y\right) \tag{3.32}$$

Proof of Lemma 3.2: See Appendix A – section A-4

Lemma 3.3: Consider a matrix $A \in \mathbb{R}^p \times \mathbb{R}^{nq}$. Let $\left[A_1 \quad \cdots \quad A_n\right]$ be a partition of A with $A_i \in \mathbb{R}^p \times \mathbb{R}^q$. We have

$$\left(I_n \otimes vec^T(A)\right)\left(vec\left(I_n\right) \otimes I_{pq}\right) = mat^T_{pq \times n}\left(vec(A)\right) \tag{3.33}$$

Proof of Lemma 3.3: See Appendix A – section A-5.

In this section, we have introduced the main notations and proprieties related to the Kronecker matrix product and vector power tensor. In particular, new results have been proposed with their proofs attached in Appendix A..

3.4 Vector power series motivation and multivariable Taylor expansion

Since certain functions are hard to implement in practice, the art of approximation becomes a mathematical solution to overcome this problem. One of the techniques is the polynomial approximation based on Taylor series development. These series allow the representation of a function as an infinite sum of terms that are calculated from the values of the function and its derivatives about a given point [36].

In the following, we introduce the Taylor series of any multivariable function. Then, we investigate the representation of particular mono-variable and two-variable functions. Their performances motivate the use of these series in terms of approximation.

Let $x=(x_1 \quad \cdots \quad x_n)^T$ be a vector of dimension n, and $f(x)$ a function of the vector x. The Taylor series development of $f(x)$ about the point $a=(a_1 \quad \cdots \quad a_n)^T$ is defined by [36]

$$f(x)=\sum_{k_1\geq 0}\sum_{k_2\geq 0}\cdots\sum_{k_n\geq 0}\prod_{j=1}^{n}\frac{\left(x_j-a_j\right)^{k_j}}{k_j\,!}\frac{\partial^{k_j} f}{\partial x_j^{k_j}}(a) \tag{3.34}$$

The Taylor series representation is often used to approximate a function by a finite number of terms so called Taylor polynomial. In vector calculus, consider f a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, *i.e.*, given by m real-valued component functions f_j ; $j=1,\cdots,m$ in $x=\left(x_1,\cdots,x_n\right)^T$. A particular case of the Taylor development of the multi-variable vector function $f(x)$, truncated at order 1 about a , represents simply its linearization about a and is given by

$$f(x)=f(a)+Df(a)(x-a) \tag{3.35}$$

where $Df(a)$ the Jacobian matrix of f , evaluated at $x=a$, obtained by the computation of the potential derivatives of all components of f at a , as follows

$$Df(a)=\begin{pmatrix}\frac{\partial f}{\partial x_1}(a)\cdots & \frac{\partial f}{\partial x_n}(a)\end{pmatrix} \tag{3.36}$$

$$=\begin{pmatrix}\frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a)\\ \vdots & & \vdots\\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a)\end{pmatrix} \tag{3.37}$$

A particular case of any multivariable function $f(x)\in\mathbb{R}$ is the Taylor series expansion of the second order that can be written as

$$f(x) \approx f(a) + Df(a)\cdot(x-a) + \frac{1}{2}(x-a)^T D^2 f(a)(x-a) \tag{3.38}$$

where $Df(a)$ is the gradient of the real valued function f evaluated at $x=a$, given by

$$Df(a) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(a) \cdots & \frac{\partial f}{\partial x_d}(a) \end{pmatrix} \tag{3.39}$$

and, $D^2 f(a)$ is the Hessian Matrix of f evaluated at $x=a$, given by

$$D^2 f(a) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(a) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(a) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) & \frac{\partial^2 f}{\partial x_2^2}(a) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(a) & \frac{\partial^2 f}{\partial x_n \partial x_1}(a) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(a) \end{bmatrix} \tag{3.40}$$

In order to illustrate this approximation, we consider in Appendix B examples of scalar, two variable functions and also dynamics. We present for each example the mathematical approximation about the origin for different orders of truncations. And then, we compare the plots of the original function with the approximated ones in order to show the performances in terms of curve fitting and domain of attraction.

3.5 Conclusion

In this chapter, we presented the KP and VPS algebra. We begin with an introduction in section 3.1. Then, in section 3.2, we presented some definitions related to the KP algebra. In section 3.3, we introduced some KP proprieties using given and new theorems and lemmas. The proofs of these new results are presented in Appendix A. The VPS formulation is presented in section 3.4 using multivariable Taylor expansion. Then, we conclude this chapter. The motivation for such representation of nonlinearities is detailed in Appendix B using examples of algebraic functions and dynamic systems.

4 Optimal control theory

4.1 Introduction

Optimization is the act of obtaining the best result under given circumstances. In design, construction and maintenance of any engineering system, engineers have to take many technological and managerial decisions at several stages. The ultimate goal of all such decisions is either to minimize the effort required or to maximize the desired benefit. Since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables, the optimization can be defined as the process of finding the condition giving the maximum or minimum value of a certain function [38].

The applications of optimization in engineering are various and wide. In the following, we depict some of these applications [38] that are still recorded:

- Design of aircraft and aerospace structure for minimum weight;
- Finding the optimal trajectories of space vehicles;
- Optimum design of linkages, cams, gears, machine tools and other mechanical components;
- Optimal production planning, controlling and scheduling;
- Optimum design of control system.

Optimization problems can be classified in several ways based on

- The existence of constraints;
- The nature of the design variables;
- The physical structure of the problem;
- The nature of the equations involved.

In this chapter, we will be limited to the study of the optimization problems based on the existence or no of the constraint variables. In section 4.2, we consider the problem of optimization with no constraints. In Section 4.3, we analyze the optimization problem with equality constraints based on the method of Lagrange multipliers. And finally, in Section 4.4, we present the optimal control problem in general, then the case of the infinite horizon and the linear quadratic regulator (LQR) problems.

4.2 Free constraint optimization

In this section, we state different mathematical programming problems of unconstrained optimization.

4.2.1 Multivariable function optimization without constraint

The unconstrained optimization problem can be defined as

Find

$$x = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \tag{4.1}$$

which minimizes $f(x)$, where $x \in \mathbb{R}^n$ is called the decision vector and $f(x)$ is a real valued function called the objective function. Without loss of generality, this optimization refers to a minimization since the maximum of a function can be found by inverting the minimum of the negative of the same function [38]. Such a minimization problem needs necessary and sufficient conditions to be fulfilled. These conditions will be discussed in the following section.

4.2.2 Functional minimization without constraint

To solve the problem of unconstrained minimization, we consider the necessary and sufficient conditions for the minimum or maximum of multi-variable function $f(x)$ given by following theorems [38].

Theorem 4.1 (Necessary Condition): If $f(x)$ has a maximum or minimum at $x = x^*$, and if $\left.\frac{\partial f}{\partial x_i}\right|_{x=x^*}$ exist, then

$$\left.\frac{\partial f}{\partial x_i}\right|_{x^*} = 0, \quad \forall i = 1 \cdots n \tag{4.2}$$

(4.2) is equivalent to

$$Df\left(x^*\right)=\begin{pmatrix}\dfrac{\partial f}{\partial x_1}\left(x^*\right) & \cdots & \dfrac{\partial f}{\partial x_1}\left(x^*\right)\end{pmatrix} \tag{4.3}$$

where $Df\left(x^*\right)^T$ is the gradient of the real valued function $f\left(x\right)$.

Theorem 4.2 (Sufficient Condition): Given x^* an extreme point (minimum or maximum), we denote by $D^2 f\left(x^*\right)$ the Hessian of the real valued function $f\left(x\right)$. The Hessian matrix correspond to the second partial derivative of $f\left(x\right)$.

$$D^2 f\left(x^*\right)=\left.\begin{pmatrix}\dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n}\\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_2}\\ \vdots & \vdots & \ddots & \vdots\\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2}\end{pmatrix}\right|_{x=x^*} \tag{4.4}$$

If $D^2 f\left(x^*\right)$ is positive definite, then x^* is a relative minimum. And, if $D^2 f\left(x^*\right)$ is negative definite then x^* is a relative maximum.

The proofs of these two theorems are commonly reported in the literature (see for example [38]).

4.2.3 Calculus of variations-problem statement and solution

The calculus of variations is the problem dealing with the determination of extrema (maxima and minima) of a functional, where the functional can be defined as a function of several other functions. In particular, the calculus of variations can be used to solve trajectory optimization problems [38]. A well-known problem of the calculus of variations with no constraints is the mathematical programming optimization of an integral amount given by [38].

Find a function $u(x)$ that minimizes the functional (integral)

$$J = \int_{x_1}^{x_2} F\left(x, u(x), \frac{du}{dx}(x) \right) dx \tag{4.5}$$

where J and F are considered functionals and x is an independent variable in the interval $[x_1, x_2]$.

In the following, we denote by $u^{'} = \frac{du}{dx}(x)$ the first derivative of u with respect to x. Let $u(x_1) = u_1$ and $u(x_2) = u_2$ be the boundary conditions of the problem (4.8). The calculus of variations is the mathematical procedure used to select the correct solution from a number of tentative solutions [52-53]. Any tentative solution $\bar{u}(x)$ in the neighbourhood of the exact solution $u(x)$ may be represented

$$\bar{u}(x) = u(x) + ¢u(x) \tag{4.6}$$

where variation $¢u(x)$ is an infinitesimal. It's considered as an arbitrary change in u for a fixed value of the variable x (*i.e.*, for $¢x = 0$). Note that the operation of variation is commutative with respect to both the integration and the differentiation, that is,

$$¢\left(\int F dx\right) = \int (¢F) dx \tag{4.7}$$

and

$$¢\left(\frac{du}{dx}\right) = \frac{d}{dx}(¢u) \tag{4.8}$$

We define the variation of the function $F(x, u, u^{'})$ introduced in (4.8) as follows

$$¢F = \frac{\partial F}{\partial x} ¢x + \frac{\partial F}{\partial u} ¢u + \frac{\partial F}{\partial u^{'}} ¢u^{'} \tag{4.9}$$

For a fixed value of the variable x, (4.12) is reduced to

$$¢F = \frac{\partial F}{\partial u} ¢u + \frac{\partial F}{\partial u^{'}} ¢u^{'} \tag{4.10}$$

Then, using (4.10), the variation of the functional J, given by (4.8) is obtained. If we consider the condition of the stationariness of J, we write the necessary condition, $¢J = 0$, as the vanishing of first derivative of J (similarly to the maximization or minimization of simple functions in ordinary calculus).

$$¢J = \int_{x_1}^{x_2} ¢F dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial u} ¢u + \frac{\partial F}{\partial u^{'}} ¢u^{'} \right) dx \tag{4.11}$$

By integrating the second and third terms by parts, we obtain

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial u^{'}} ¢u^{'} dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial u^{'}} ¢\left(\frac{\partial u}{\partial x} \right) dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial u^{'}} \frac{\partial}{\partial x} (¢u)\, dx = \frac{\partial F}{\partial u^{'}} ¢u \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial F}{\partial u^{'}} \right) ¢u\, dx \tag{4.12}$$

Substitute (4.12) in (4.11)

$$¢J = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial u} - \frac{d}{dx} \left(\frac{\partial F}{\partial u^{'}} \right) \right] ¢u\, dx + \left[\frac{\partial F}{\partial u^{'}} \right] ¢u \Big|_{x_1}^{x_2} \tag{4.13}$$

with $¢u^{'}(x_1) = 0, \quad ¢u^{'}(x_2) = 0$. Since $¢u$ is arbitrary, each term of (4.13) must vanish individually, *i.e.*,

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial u} - \frac{d}{dx} \left(\frac{\partial F}{\partial u^{'}} \right) \right] ¢u ¢x = 0 \tag{4.14}$$

and

$$\left[\frac{\partial F}{\partial u^{'}} \right] ¢u \Big|_{x_1}^{x_2} = 0 \tag{4.15}$$

According to the fundamental lemma of calculus of variations [53], the part of integrand in brackets is zero, *i.e.*,

$$\frac{\partial F}{\partial u} - \frac{d}{dx} \left(\frac{\partial F}{\partial u^{'}} \right) = 0 \tag{4.16}$$

Equation (4.16) called Euler-Lagrange equation is the governing differential equation for the given problem. From (4.15), we obtain the boundary conditions.

$$\left[\frac{\partial F}{\partial u^{'}}\right]\Bigg|_{x_1}^{x_2} = 0 \qquad (4.17)$$

and

$$\left[\frac{\partial F}{\partial u^{''}}\right]\Bigg|_{x_1}^{x_2} = 0 \qquad (4.18)$$

(4.17) and (4.18) are called natural boundary condition or free boundary conditions. If these natural boundary conditions are not satisfied, we should have

$$¢u(x_1) = 0\,,\quad ¢u(x_2) = 0,\; ¢u^{'}(x_1) = 0 \; and \; ¢u^{'}(x_2) = 0 \qquad (4.19)$$

in order to satisfy equations (4.15). (4.19) are called geometric or forced boundary conditions. The Euler Lagrange equation (4.16) is a necessary, but not sufficient condition for the minimization problem (4.5).

4.3 Optimization with equality constraints

In this section, we discuss different techniques of constrained optimization problems. We will be limited to the case of equality constraints.

4.3.1 Multivariable function optimization with constraint

Given multivariable function f and g_i, $j = 1, 2, \ldots, p$, the optimization problem with equality constraints can be defined as [38]

$$\text{Find } X = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(X) \qquad (4.20)$$

Subject to

$$g_j(X)=0 \quad , \; j=1,2,\cdots,n \tag{4.21}$$

In the literature, we find different methods that are developed to solve this problem. We depict the direct substitution, the constraint variation and the Lagrange multipliers.

4.3.2 Solving the minimization with equality constraints

The first method to solve the problem of function minimization which is the direct substitution is very intuitive. It consists in solving simultaneously the n equality constraints. Then, we express any set of m variables in terms of the remaining $n-m$ variables. These expressions are substituted into the original objective function which results in a new objective in only $n-m$ variables with no constraints. Then, the optimum can be found by using the techniques of the unconstrained optimization discussed above. This method is a simple (theoretical) method but not convenient in practice, because the constraint equations are often nonlinear and hard to solve [38].

The second method, the constraint variation, consists in setting the total differential of the objective function equal to zero and then developing the Taylor expression of the constraint function about the minimum point and deducing the variation in $dx_1,\ldots,dx_n$. Then, substituting these variations in the main equation leads to the necessary condition for the constrained optimization [38].

The third method of Lagrange multipliers will be the subject of the next subsection.

4.3.3 Method of Lagrange multipliers

This method will be the first introduced in a simple case of a two variable minimization problem subject to one equality constraint (*i.e.*, $n=2$ and $m=1$). Then, we present the problem of a functional minimization in the case of one dependent variable. Finally, we state the general form of a functional minimization using the Lagrange multiplier formulation. This technique will be the key element of the optimal control theory that will be discussed latter in the next section.

4.3.3.1 Case of two variables and one equality constraint

We consider the particular case of an objective function of two variables f with one equality constraint g. Hence, the problem formulation of this case is to

Minimize $f(x_1, x_2)$ such that

$$g(x_1, x_2) = 0 \tag{4.22}$$

Using Theorem 4.1, introduced in section 4.2, we note the necessary condition for the existence of the minimum at $\left(x_1^*, x_2^*\right)$ is

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0 \tag{4.23}$$

From the constraint

$$g\left(x_1^*, x_2^*\right) = 0 \tag{4.24}$$

we write the new constraint

$$g\left(x_1^* + dx_1, x_2^* + dx_2\right) = 0 \tag{4.25}$$

The variations dx_1 and dx_2 verifying (4.25) about the point $\left(x_1^*, x_2^*\right)$ are called admissible variations [38]. The constraint (4.25) is now rewritten using Taylor's series expansion of g about $\left(x_1^*, x_2^*\right)$

$$g\left(x_1^*, x_2^*\right) + \frac{\partial}{\partial x_1}\left(x_1^*, x_2^*\right) dx_1 + \frac{\partial}{\partial x_2}\left(x_1^*, x_2^*\right) dx_2 = 0 \tag{4.26}$$

Since $g\left(x_1^*, x_2^*\right) = 0$, we obtain at $\left(x_1^*, x_2^*\right)$

$$dg = \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 = 0 \tag{4.27}$$

By assuming that $\frac{\partial g}{\partial x} \neq 0$, at $\left(x_1^*, x_2^*\right)$, (4.27) can be re-written as

$$dx_2 = -\frac{\partial g / \partial x_1}{\partial g / \partial x_2} dx_1 \tag{4.28}$$

We substitute (4.28) in (4.23)

$$\left(\frac{\partial f}{\partial x_1} - \frac{\partial g / \partial x_1}{\partial g / \partial x_2} \frac{\partial f}{\partial x_2} \right) \cdot dx_1 = 0 \tag{4.29}$$

at $\left(x_1^*, x_2^*\right)$, for all admissible variations dx_1 chosen arbitrary, that is

$$\frac{\partial f}{\partial x_1} - \frac{\partial f / \partial x_2}{\partial g / \partial x_2} \frac{\partial g}{\partial x_1} = 0 \tag{4.30}$$

at $\left(x_1^*, x_2^*\right)$. (4.30) represents the necessary condition for the existence of the minimum at $\left(x_1^*, x_2^*\right)$ used in the method of constrained variations [38]. We denote by a

$$a = -\left.\frac{\partial f / \partial x_2}{\partial g / \partial x_2}\right|_{\left(x_1^*, x_2^*\right)} \tag{4.31}$$

the Lagrange multiplier [38]. By substituting (4.31) in (4.30) and also rewriting (4.31), we obtain [38]

$$\left.\left(\frac{\partial f}{\partial x_1} + a \frac{\partial g}{\partial x_1} \right)\right|_{\left(x_1^*, x_2^*\right)} = 0 \tag{4.32}$$

and

$$\left(\frac{\partial f}{\partial x_2} + a \frac{\partial g}{\partial x_2}\right)\Bigg|_{\left(x_1^*, x_2^*\right)} = 0 \tag{4.33}$$

Note that (4.28), (4.32) and (4.33) represent the necessary conditions for x^* to be an extreme point of the problem (4.23). These conditions are treated by defining the Lagrange function L as [38]

$$L\left(x_1, x_2, a\right) = f\left(x_1, x_2\right) + a\, g\left(x_1, x_2\right) \tag{4.34}$$

In fact, the derivatives of the Lagrange function with respect to the variables x_1, x_2 and a, respectively, at the extreme point $\left(x_1^*, x_2^*\right)$ lead to (4.28), (4.32) and (4.33). The sufficient condition for (4.23) to have a minimum at $\left(x_1^*, x_2^*\right)$ is that the quadratic amount Q, defined by

$$Q = \frac{\partial^2 L}{\partial x_1^2} dx_1^2 + 2\frac{\partial^2 L}{\partial x_1 \partial x_2} dx_1 dx_2 + \frac{\partial^2 L}{\partial x_2^2} dx_2^2 \tag{4.35}$$

evaluated at $\left(x_1^*, x_2^*\right)$, is positive definite for all values of the admissible variations dx_1 and dx_2 (see theorem 2.6 of [38]. According to [38], if Q is positive definite for all dx_1 and dx_2, then all the roots of the following equation are positive

$$\begin{vmatrix} L_{11} - z & L_{12} & g_1 \\ L_{21} & L_{22} - z & g_2 \\ g_1 & g_2 & 0 \end{vmatrix} = 0 \tag{4.36}$$

where

$$L_{11} = \frac{\partial^2 L}{\partial x_1^2}\Bigg|_{\left(x_1^*, x_2^*\right)},\ L_{12} = L_{21} = \frac{\partial^2 L}{\partial x_1 \partial x_2}\Bigg|_{\left(x_1^*, x_2^*\right)},\ L_{22} = \frac{\partial^2 L}{\partial x_2^2}\Bigg|_{\left(x_1^*, x_2^*\right)} \tag{4.37}$$

and

$$g_1 = \left.\frac{\partial g}{\partial x_1}\right|_{\left(x_1^*, x_2^*\right)}, \; g_2 = \left.\frac{\partial g}{\partial x_2}\right|_{\left(x_1^*, x_2^*\right)} \tag{4.38}$$

Note that (4.36) is affine equation in z .

4.3.3.2 Case of functional minimization – Example of one dependant variable

In the case of minimization of an integral functional, the problem will be formulated as follows [38]:

Find $u(x)$ which minimizes

$$J = \int_{x_1}^{x_2} F\left(x, u, \frac{du}{dx}\right) dx \tag{4.39}$$

Subject to the constraint

$$G\left(x, u, \frac{du}{dx}\right) = 0 \tag{4.40}$$

Where G may be an integral function too.

4.3.3.3 Case of functional minimization – General form

The general problem of minimization of integral functional can be formulated as follows [38]:

Find the set of n functions $u_1(x, y, z), u_2(x, y, z), \cdots, u_n(x, y, z)$ in the dependent variables x, y, z which make the functional

$$J = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_3} F\left(x, y, z, u_1, \ldots, u_n, \frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_n}{\partial x}, \frac{\partial u_1}{\partial y}, \ldots, \frac{\partial u_n}{\partial y}, \frac{\partial u_1}{\partial z}, \ldots, \frac{\partial u_n}{\partial z}\right) dxdydz \tag{4.41}$$

stationary, subject to m constraints

$$G_j\left(x, y, z, u_1, \ldots, u_n, \frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_n}{\partial x}, \frac{\partial u_1}{\partial y}, \ldots, \frac{\partial u_n}{\partial y}, \frac{\partial u_1}{\partial z}, \ldots, \frac{\partial u_n}{\partial z}\right) = 0 \tag{4.42}$$

for $j = 1, \ldots, m$. The Lagrange multiplier method consists of minimizing the functional [38]

$$L = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \left(F + \alpha_1 G_1 + \ldots + \alpha_m G_m \right) dxdydz \tag{4.43}$$

where α_i, $i = 1, \ldots, m$, are the Lagrange multipliers and functions of x, y and z. This general formulation of the minimization of an integral functional using the Lagrange multipliers represents the key element of the optimal control theory. The latter will be the subject of the next section.

4.4 Optimal control theory

4.4.1 General problem

Given a vector $x = \left(x_1, \cdots, x_n \right)^T \in \mathbb{R}^n$, the optimal control problem can be formulated as follows [40-41]:

Find the vector $u = \left(u_1, \cdots, u_m \right)^T \in \mathbb{R}^m$, which minimizes the functional, called performance index

$$J = \int_0^T g\left(x, u, t \right) dt \tag{4.44}$$

subject to the constraint

$$\dot{x} = f\left(x, u, t \right) \tag{4.45}$$

with the boundary condition $x(0) = x_0$. u designates the input vector, x the state vector and t the time. The necessary condition for the general problem is

$$f_i\left(x, u, t \right) - \dot{x}_i = 0 \tag{4.46}$$

We introduce a Lagrange multiplier a_i, also known as the adjoint variable, for the i^{th} constraint equation. Based on Lagrange multiplier method, the augmented functional is given by [41]

$$L = \int_0^T \left[g + \sum_{i=1}^{n} a_i \left(f_i - \dot{x}_i \right) \right] dt \tag{4.47}$$

In the following, we use the Hamiltonian function H, defined from (4.47)

$$H\left(x,u,a,t\right) = g + \sum_{i=1}^{n} a_i f_i = f_0\left(x,u,t\right) + a^T f\left(x,u,t\right) \tag{4.48}$$

with $a = \left(a_1,\cdots,a_n\right)^T$. From (4.48), (4.47) becomes

$$L = \int_0^T \left(H - \sum_{i=1}^{n} a_i \dot{x}_i \right) dt = \int_0^T \left(H - a^T \dot{x} \right) dt \tag{4.49}$$

The integrant

$$F = H - a^T \dot{x} = H - \sum_{i=1}^{n} a_i \dot{x}_i \tag{4.50}$$

depends on x, u and t [38]. Hence using (4.50), the Euler Lagrange equations corresponding to the functional L of (4.82) become

$$\frac{\partial F}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i}\right) = \frac{\partial\left(H - \sum_{i=1}^{n} a_i \dot{x}_i\right)}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial\left(H - \sum_{i=1}^{n} a_i \dot{x}_i\right)}{\partial \dot{x}_i}\right) = 0 \quad ; i = 1,\cdots,n \tag{4.51}$$

and

$$\frac{\partial F}{\partial u_j} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}_j}\right) = \frac{\partial\left(H - \sum_{i=1}^{n} a_i \dot{x}_i\right)}{\partial u_j} - \frac{d}{dt}\left(\frac{\partial\left(H - \sum_{i=1}^{n} a_i \dot{x}_i\right)}{\partial \dot{u}_j}\right) = 0 \quad ; j = 1,\cdots,m \tag{4.52}$$

Or equivalently,

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \tag{4.53}$$

and

$$\frac{\partial F}{\partial u} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}}\right) = 0 \tag{4.54}$$

Since $\frac{\partial \dot{x}_i}{\partial x_i} = 0$, $\frac{\partial \dot{x}_i}{\partial \dot{x}_i} = 1$, $\frac{\partial \dot{x}_i}{\partial u_j} = 0$ and $\frac{\partial \dot{x}_i}{\partial \dot{u}_j} = 0$, we obtain [42]

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i} \quad \forall i = 1, \cdots, n \tag{4.55}$$

and

$$\frac{\partial H}{\partial u_j} = 0 \quad \forall j = 1, \cdots, m \tag{4.56}$$

Or equivalently,

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \tag{4.57}$$

and

$$\frac{\partial H}{\partial u} = 0 \tag{4.58}$$

The optimum solution for x, u and λ can be obtained by solving (4.45), (4.57) and (4.58) in the unknown $n\, x_i$'s, $n\, \lambda_i$'s, $i = 1, \ldots, n$, and $n\, u_j$'s, $j = 1, \ldots, m$, unknowns. These equations are called the canonical Hamilton equations [42]. The solution of these equations will contain $2n$ constants of integration. To determine these constants we need n equations of the initial conditions

$$x^*(t_0) = x_0 \tag{4.59}$$

and also additional n, or $n+1$, conditions depending on whether or not T is specified [41]. The set of all these conditions refers to the boundary conditions. Different cases arise. First, we depict problems with fixed final time T (*i.e.*, the final time T is specified) and problems with free final time T (*i.e.*, the final time T is free).

4.4.1.1 Problems with fixed final time

We depict [41]

- The problem of final state specified (*i.e.*, T and $x(T)$ are specified);
- The problem of final state free (*i.e.*, T is specified and $x(T)$ is free);
- The problem of final state lying on a surface defined by $s(x(T))=0$.

4.4.1.2 Problems with free final time

We depict also different situations [41]

- The problem of final state fixed (*i.e.*, T is free and $x(T)$ fixed);
- The problem of final state free (*i.e.*, T and $x(T)$ are free);
- The problem of final state moving with $p(T)$ (*i.e.*, $x^*(T)=p(T)$);
- The problem of final state lying on the surface defined by $s(x(T))=0$;
- The problem of final state lying on a moving surface defined by $s(x(T),T)=0$.

The different scenarios listed above are set in the different boundary condition equations (4.21) and (4.23). In [41], the author has stated the general form of these boundary conditions, corresponding to the optimization problem (4.44) and (4.45) for all $t\in[0,T]$, as follows:

$$ª(T)^T \cdot ¢x(T) = H(x(T),u(T),ª(T),T)\cdot ¢T \quad (4.60)$$

In the following, we consider exclusively the case of specified final time T and free state $x(T)$ leading to the investigation of the problem of infinite horizon (*i.e.*, $T=\infty$) discussed later. Then the substitutions in (4.60) are

$$¢T = 0 \text{ and } ¢x(T) \text{ any variation} \quad (4.61)$$

Thus, we obtain the specific boundary condition relationships [41]

$$x(0) = x_0 \text{ and } ª(T) = 0 \tag{4.62}$$

4.4.2 Hamilton-Jacobi equation

In this section, we treat the problem of the optimal control with the finite horizon case (*i.e.*, T is finite) which leads to the well-known HJE [39].

Consider a nonlinear system defined by the dynamics

$$\dot{x} = f(x,u,t) \tag{4.63}$$

with the initial condition

$$x(t_0) = x_0 \tag{4.64}$$

subject to the functional cost to be minimized

$$J(u) = \int_0^T g(x,u,t)dt \tag{4.65}$$

The problem is to find an admissible control u^* that forces the system (4.63) to follow an admissible trajectory x^* that minimizes the performance (4.65). The initial time t_0 and the initial states $x(t_0)$ are specified [41]. Without loss of generality, the initial instant is reduced to $t_0 = 0$. We define the optimal cost $V(x,t)$ with an initial state $x(t)$ at t by [5-39]

$$V(x,t) = \min_u \int_t^T g(x,u,²) \tag{4.66}$$

i.e.,

$$V(x,t) = \int_t^T g(x,u^*,²) \tag{4.67}$$

where $u^*(t)$ is the optimal control. From (4.66), we set if the system starts from $x(t)$ at t, then $V(x,t) = \min_u J$. $V(x,t)$ is independent of u because if the initial state $x(t)$ and its time t are specified, then the particular control u is "abstractly" determined and minimizes $V(x,t)$ [39-40]. So, to find the optimal control u^* which minimizes (4.65), and then, $V(x(0),0)$ for various $x(0)$, we can start by the evaluation of (4.67) for all t and $x(t)$, and then the associated optimal control u^* as follows [39]:

Given $t \in [0,T]$ and $t_i \in [t,T]$, we have

$$V(x(t),t) = \min_{u(s),s.t.\,s\in[t,T]} \int_t^T g(x,u,\tau)d\tau \tag{4.68}$$

$$= \min_{u(s),s.t.\,s\in[t,t_i]} \left[\min_{u(s),s.t.\,s\in[t,T]} \left(\int_t^{t_i} g(x,u,\tau)d\tau + \int_{t_i}^T g(x,u,\tau)d\tau \right) \right] \tag{4.69}$$

u^* is obtained by the concatenation of $u(s)$ with $s \in [t,t_i]$ and $s \in [t_i,T]$. Note that the term $\int_t^{t_i} g(x,u,\tau)d\tau$ is independent of $u(s)$ for $s \in [t_i,T]$ and

$$\int_{t_i}^T g(x,u,\tau)d\tau = V(x(t_i),t_i) \tag{4.70}$$

Then,

$$V(x,t) = \min_{u(s),s.t.\,s\in[t,t_i]} \left[\int_t^{t_i} g(x,u,\tau)d\tau + V(x(t_i),t_i) \right] \tag{4.71}$$

We set $t_i = t + \Delta t$, where Δt is small. We apply the Taylor's theorem to expand the right hand side of (4.71) at the first order

$$V(x,t) = \min_{u(s),s.t.\,s\in[t,t+\Delta t]} \left[\int_t^{t+\Delta t} g(x,u,\tau)d\tau + V(x(t+\Delta t),t+\Delta t) \right] \tag{4.72}$$

We introduce $G(x,u,t)$ such that $\frac{dG}{dt} = g(x,u,t)$. Hence, the first order Taylor expansion of $\int_t^{t+\Delta t} g(x,u,\tau)d\tau$ is

$$\int_t^{t+\Delta t} g(x,u,\tau)d\tau = G(x(t+\Delta t),u(t+\Delta t),t+\Delta t) - G(x(t),u(t),t) \tag{4.73}$$

Also, we have

$$G\left(x(t+\Delta t),u(t+\Delta t),t+\Delta t\right)=G\left(x(t),u(t),t\right)+\frac{dG}{dt}\Delta t \tag{4.74}$$

that is,

$$G\left(x(t+\Delta t),u(t+\Delta t),t+\Delta t\right)-G\left(x(t),u(t),t\right)=\frac{dG}{dt}\Delta t=g(x,u,t)\Delta t \tag{4.75}$$

Then, from (4.73) and (4.75) we obtain

$$\int_t^{t+\Delta t} g(x,u,\tau)\,d\tau = g(x,u,t)\Delta t \tag{4.76}$$

Also, we use the first order Taylor expansion of $V\left(x(t+\Delta t),t+\Delta t\right)$

$$V\left(x(t+\Delta t),t+\Delta t\right)=V\left(x(t),t\right)+\frac{dV\left(x(t),t\right)}{dt}\Delta t \tag{4.77}$$

Noting,

$$\frac{dV}{dt}\left(x(t),t\right)=\left(\frac{\partial V}{\partial x}\left(x(t),t\right)\right)^T\frac{\partial x}{\partial t}+\frac{\partial V}{\partial t} \tag{4.78}$$

and

$$V\left(x(t+\Delta t),t+\Delta t\right)=V\left(x(t),t\right)+\left[\left(\frac{\partial V}{\partial x}\left(x(t),t\right)\right)^T\frac{\partial x}{\partial t}+\frac{\partial V}{\partial t}\right]\Delta t \tag{4.79}$$

we substitute (4.145) into (4.138)

$$V(x,t)=\min_{u(s),s.t.\,s\in[t,\Delta t]}\left[g(x,u,t)\Delta t+V\left(x(t),t\right)+\left(\left(\frac{\partial V}{\partial x}\left(x(t),t\right)\right)^T\dot{x}+\frac{\partial V\left(x(t),t\right)}{\partial t}\right)\Delta t\right] \tag{4.80}$$

As $V(x,t)$ is independent of u, then

$$V(x,t)=V(x(t),t)+\min_{u(s),s.t.\,s\in[t,\Delta t]}\left[\left(g(x,u,t)+\left(\frac{\partial V}{\partial x}\right)^T\dot{x}+\frac{\partial V}{\partial t}\right)\Delta t\right] \quad (4.81)$$

Then,

$$\min_{u(s),s.t.\,s\in[t,\Delta t]}\left[\left(g(x,u,t)+\left(\frac{\partial V}{\partial x}\right)^T\dot{x}+\frac{\partial V}{\partial t}\right)\Delta t\right]=0 \quad (4.82)$$

And, using (4.63), we write from (4.82)

$$\frac{\partial V(x,t)}{\partial t}=-\min_{u(s),s.t.\,s\in[t,\Delta t]}\left[g(x,u,t)+\left(\frac{\partial V}{\partial x}\right)^T f(x,u,t)\right] \quad (4.83)$$

As Δt approaches zero, we obtain [39]

$$\frac{\partial V(x,t)}{\partial t}=-\min_{u(t)}\left[g(x,u,t)+\left(\frac{\partial V}{\partial x}\right)^T f(x,u,t)\right] \quad (4.84)$$

The equation (4.83) is the HJE. This equation becomes

$$\frac{\partial V(x,t)}{\partial t}=-g(x,u^*,t)-\left(\frac{\partial V}{\partial x}\right)^T f(x,u^*,t) \quad (4.85)$$

where u^* is the optimal control expression given by [39]

$$u^*=\arg\left(\min_{u(t)}\left[g(x,u,t)+\left(\frac{\partial V}{\partial x}\right)^T f(x,u,t)\right]\right) \quad (4.86)$$

4.4.3 Infinite horizon

In this section, we introduce the infinite horizon optimal control problem for any general time invariant nonlinear system. Then, we discuss the case of Linear Time Varying systems (LTV). And finally, we investigate the case of LTI systems known as the LQR problem.

4.4.3.1 Case of time invariant systems

Consider the time invariant nonlinear dynamics

$$\dot{x} = f(x,u) \tag{4.87}$$

subject to the time performance index

$$J(u) = \int_0^\infty g(x,u)dt \tag{4.88}$$

with $x(0) = x_0$. Then, for the stationary and infinite time, *i.e.*, $T = \infty$, we have [3]

$$\frac{\partial V}{\partial t}(x,t) = 0 \tag{4.89}$$

As V is explicitly independent of t. Then, we obtain the equivalent HJE

$$g(x,u^*) + \left(\frac{\partial V}{\partial x}\right)^T f(x,u^*) = 0 \tag{4.90}$$

and

$$u^* = \arg\left(\min_u \left[g(x,u) + \left(\frac{\partial V}{\partial x}\right)^T f(x,u)\right]\right) \tag{4.91}$$

i.e.,

$$\frac{\partial}{\partial u}\left[g(x,u) + \left(\frac{\partial V}{\partial x}\right)^T f(x,u)\right]\Bigg|_{u=u^*} = 0 \tag{4.92}$$

4.4.3.2 Case of linear time varying systems

Consider the system

$$\dot{x} = F(t)x(t) + G(t)u(t) \tag{4.93}$$

with $x(t_0) = x_0$. We assume that the matrices $F(t)$ and $G(t)$ are continuous field functions. Given the weighting matrices $Q(t)$ and $R(t)$ are continuous, symmetric, nonnegative, and positive definite, respectively. We define the performance index

$$J = \frac{1}{2}\int_0^\infty \left(x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \right) dt \tag{4.94}$$

The optimal control law $u^*(t)$, which minimizes the performance index J, satisfies the following Theorem.

Theorem 4.1: Let's consider the system given by the dynamics (4.93) subject the performance index (4.94). It follows [39]

i) $\tilde{P}(t) = \lim_{T\to\infty} P(t,T)$ exists, where $P(t,T)$ is the solution of the equation $PF + PF^T - PGR^{-1}G^T P + Q = -\dot{P}$.

ii) $x^T(t)\tilde{P}(t)x(t)$ is the optimal performance index.

iii) $u^*(t) = -R^{-1}G^T(t)\tilde{P}(t)x(t)$ is the optimal control law.

We note that the system (4.93) is completely controllable for every time t. If, given an arbitrary state $x(t)$ at time t, there exists a control depending on $x(t)$ at t and a time t_2 depending on t such that application of this control over the interval $[t_1\ t_2]$ takes the state $x(t)$ to the zero state at time t_2.

The proof of this theorem is discussed with details in [39].

4.4.4 LQR problem

In the following, we solve the problem of LQR for the LTI system, and we show how this problem leads to the resolution of the well-known algebraic Ricatti equation (ARE). Therefore, we discuss the stability of the closed loop optimal control.

4.4.4.1 Optimal control design

Let's consider the linear system

$$\dot{x} = Ax + Bu \tag{4.95}$$

with $x(0) = x_0$. The performance index associated with (4.167) is defined as

$$J = \frac{1}{2}\int_0^{\infty}\left(x^T Qx + u^T Ru + 2x^T Nu\right)dt \tag{4.96}$$

where Q and R are symmetric matrices. Q is non-negative definite, whereas R is positive definite. N a matrix of appropriate dimensions. The task will be to design a stabilizing linear state-feedback controller of the form $u = -Kx$ which minimizes the performance index J . This optimal control law will be denoted by u^*. The following presentation is based on different approaches discussed in the literature [42].

The Hamiltonian is written from (4.47)

$$H(x,u,\lambda) = \frac{1}{2}\left(u^T Ru + x^T Qx + 2x^T Nu\right) + \lambda^T (Ax + Bu) \tag{4.97}$$

From (4.57), the co-state vector $\lambda(t)$ is the solution of the vector differential equation

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} = -Qx - Nu - A^T\lambda \tag{4.98}$$

The minimization of H implies from (4.58)

$$\frac{\partial H}{\partial u} = Ru + N^T x + B^T\lambda = 0 \tag{4.99}$$

The optimal control is then obtained

$$u^*(t) = -R^{-1}N^T x - R^{-1}B^T\lambda \tag{4.100}$$

Notice $\dfrac{\partial^2 H}{\partial u^2} = R > 0$ which leads to the minimum of (4.96). Substituting (4.100) into (4.95) and (4.98) to obtain

$$\dot{x} = \left(A - BR^{-1}N^T\right)x - BR^{-1}B^T \text{ª} \tag{4.101}$$

and

$$\dot{\text{ª}} = \left(-Q + NR^{-1}N^T\right)x + \left(NR^{-1}B^T - A^T\right)\text{ª} \tag{4.102}$$

Or equivalently,

$$\begin{pmatrix} \dot{x} \\ \dot{\text{ª}} \end{pmatrix} = \begin{pmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & NR^{-1}B^T - A^T \end{pmatrix}\begin{pmatrix} x \\ \text{ª} \end{pmatrix} \tag{4.103}$$

Denote by $\text{¤}(t,0)$ the transition matrix associated with (4.175). A partition of $\text{¤}(t,0)$ determines a solution of (4.175) as follows

$$\begin{pmatrix} x \\ \text{ª} \end{pmatrix} = \text{¤}(t,0)\cdot\begin{pmatrix} x_0 \\ \text{ª}_0 \end{pmatrix} + \begin{pmatrix} \text{¤}_{11}(t,0) & \text{¤}_{12}(t,0) \\ \text{¤}_{21}(t,0) & \text{¤}_{22}(t,0) \end{pmatrix}\begin{pmatrix} x_0 \\ \text{ª}_0 \end{pmatrix} \tag{4.104}$$

Note from the transversality condition

$$H\left(x(T), u(T), \text{ª}(T)\right)\cdot \text{¢}T + \text{ª}(T)^T \cdot \text{¢}x(T) = 0 \tag{4.105}$$

The final time, $T \to \infty$, and the final state $x(T)$ are both free. Then, $\text{¢}T$ and $\text{¢}x(T)$ are both nonzero. Given $\text{¢}T \neq 0$ and $\text{¢}x(T) \neq 0$, from (4.123), we have

$$H\left(x(T), u(T), \text{ª}(T)\right) = 0 \text{ and } \text{ª}(T) = 0 \tag{4.106}$$

From (4.102), we write at instant T

$$\text{ª}(T) = \text{¤}_{21}(T,0)x_0 + \text{¤}_{22}(T,0)\text{ª}_0 = 0 \tag{4.107}$$

Note that $\text{¤}(t,0)$ is regular, then $\text{¤}_{22}(t,0) \neq 0 \quad \forall t \geq 0$. Thus,

$$\lambda_0 = -\Phi_{22}(T,0)^{-1}\Phi_{21}(T,0)x_0 \qquad (4.108)$$

Equivalently, using any instant t as initial time, we obtain from (4.102), $\forall t \geq 0$

$$\lambda(t) = -\Phi_{22}(t,T)^{-1}\Phi_{21}(T,t)x(t) \qquad (4.109)$$

Noting $P(t) = -\Phi_{22}(t,T)^{-1}\Phi_{21}(T,t) = -\Phi_{22}(t,T)\Phi_{21}(T,t)$, it follows $P(T) = 0$ as $\Phi(t,t) = I$, $\forall t \geq T$. We have

$$\lambda(t) = P(t)\cdot x(t) \qquad (4.110)$$

Now from (4.97), as the matrices A, B, N, Q and R are constant, we deduce that the Hamiltonian is explicitly independent of t, *i.e.*, $\frac{dH}{dt} = 0$. Then, $H\left(u(t), x(t), \lambda(t)\right)$ is constant. Finally, from the first equality of (4.106), we conclude $\forall t \geq 0$

$$H\left(x(t), u(t), \lambda(t)\right) = 0 \qquad (4.111)$$

Using (4.100) and (4.110), we rewrite (4.97) $\forall x \in \mathbb{R}^n$

$$-\frac{1}{2}x^T NR^{-1}N^T x + \frac{1}{2}x^T Qx - x^T NR^{-1}B^T P(t)x + x^T A^T P(t)x - \frac{1}{2}x^T P(t)^T BR^{-1}B^T P(t)x = 0 \qquad (4.112)$$

That is, $\forall x \in \mathbb{R}^n$

$$x^T\left[\left(A - BR^{-1}N^T\right)^T P(t) + P(t)^T\left(A - BR^{-1}N^T\right) + \left(Q - NR^{-1}N^T\right) - P(t)^T BR^{-1}B^T P(t)\right]x = 0 \qquad (4.113)$$

which reduces to

$$\left(A - BR^{-1}N^T\right)^T P(t) + P(t)^T\left(A - BR^{-1}N^T\right) + \left(Q - NR^{-1}N^T\right) - P(t)^T BR^{-1}B^T P(t) = 0 \qquad (4.114)$$

From (4.111), we have

$$\dot{a} = \dot{P}(t)x + P(t)\dot{x} \tag{4.115}$$

Using (4.103) and (4.110), we write

$$\left(-Q + NR^{-1}N^T\right)x + \left(NR^{-1}B^T - A^T\right)P(t)x = \dot{P}(t)x + P(t)\left(A - BR^{-1}N^T\right)x - P(t)BR^{-1}B^T P(t)x \tag{4.116}$$

that is,

$$\dot{P}(t) + \left(A - BR^{-1}N^T\right)^T P(t) + P(t)\left(A - BR^{-1}N^T\right) + \left(Q - NR^{-1}N\right) - P(t)BR^{-1}B^T P(t) = 0 \tag{4.117}$$

Since Q and R are symmetric, consider the transpose of both sides of (4.117) as

$$\dot{P}(t)^T + P(t)^T\left(A - BR^{-1}N^T\right) + \left(A - BR^{-1}N^T\right)^T P(t)^T + \left(Q - NR^{-1}N^T\right) - P(t)^T BR^{-1}B^T P(t)^T = 0 \tag{4.118}$$

By comparing (4.117) and (4.118), we find that both $P(t)$ and $P(t)^T$ are solutions of the same differential equation, with $P(T) = P(T)^T = 0$. From the uniqueness of solutions of differential equations, we conclude that

$$P(t) = P(t)^T \quad \forall t \geq 0 \tag{4.119}$$

Then, (4.114) becomes

$$\left(A - BR^{-1}N^T\right)^T P(t) + P(t)\left(A - BR^{-1}N^T\right) + \left(Q - NR^{-1}N^T\right) - P(t)BR^{-1}B^T P(t) = 0 \tag{4.120}$$

$P(t)$ satisfies (4.119) and (4.120) simultaneously. Thus, $\dot{P}(t) = 0$, and P is constant and solution of

$$\left(A - BR^{-1}N^T\right)^T P + P\left(A - BR^{-1}N^T\right) + \left(Q - NR^{-1}N^T\right) - PBR^{-1}B^T P = 0 \tag{4.121}$$

(4.121) is known as the ARE. The optimal control is then completed as

$$u^* = -R^{-1}\left(N + PB\right)^T x \tag{4.122}$$

4.4.4.2 Stability analysis

Consider the LTI system (4.95) subject to the minimization of the quadratic performance index (4.96). According to the general equations (4.84) and (4.85), the related steady state HJB equation is written in form [5]

$$\min_u \left(\frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + x^T N u \right) + \left(\frac{\partial V}{\partial x} \right)^T \left(Ax + Bu \right) = 0 \tag{4.123}$$

where $V(x)$ is the value of the cost function with the initial state x at t, given by (refer to the general form (4.60) [5] (and references cited therein).

$$V(x) = \min_u \int_t^{\infty} \left(\frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + x^T N u \right) d\tau \tag{4.124}$$

We note that $V(x) > 0 \; \forall x \neq 0$ if

$$\begin{pmatrix} Q & N \\ N^T & R \end{pmatrix} > 0 \tag{4.125}$$

Notice that the HJB equation (4.123) can be assimilated to the Hamiltonian equation $H(x, u, \lambda) = 0$ computed from (4.97). Then, the term $\frac{\partial V}{\partial x}$ is associated with the co-state λ given by $\lambda = Px$. Thus, we can definite $V(x)$, *s.t.*,

$$\frac{\partial V}{\partial x} = Px \tag{4.126}$$

with P solution of the ARE (4.121). Then, $V(x) = x^T P x$ is a natural Lyapunov function to determine the stability of the closed-loop system (4.95) and (4.100) with λ expressed by (4.126). In fact, we have from (4.123)

$$\dot{V} = \left(\frac{\partial V}{\partial x}\right)^T \dot{x} = \left(\frac{\partial V}{\partial x}\right)^T (Ax + Bu)$$

$$= -\frac{1}{2}x^T Qx - \frac{1}{2}u^{*T} Ru^* - x^T Nu^*$$

$$= -\frac{1}{2}\left(x^T u^{*T}\right)\begin{pmatrix} Q & N \\ N^T & R \end{pmatrix}\begin{pmatrix} x \\ u^* \end{pmatrix} \tag{4.127}$$

Using (4.125), we deduce that

$$\dot{V} < 0 \quad \forall x \neq 0 \tag{4.128}$$

Remark 4.1: By substituting (4.125) into (4.124), the positiveness of integrand of (1.124) is given by:

$$\frac{1}{2}x^T \left[Q + (N + PB)R^{-1}(N + PB)^T - 2NR^{-1}(N + PB)^T\right]x = \frac{1}{2}x^T$$
$$\left[Q + PBR^{-1}B^T P - NR^{-1}N^T\right]x > 0 \tag{4.129}$$

(4.129) holds for all $x \neq 0$ if

$$\left(Q - NR^{-1}N^T\right) + PBR^{-1}B^T P > 0 \tag{4.130}$$

Note that, using the Schur's complement [45], we have from (4.125)

$$R > 0 \text{ and } Q - NR^{-1}N^T > 0 \tag{4.131}$$

Thus, (4.130) holds if $V(x) > 0$.

Now we apply the definition of asymptotic stability in sense of Lyapunov (*i.e.*, we search if it exists a Lyapunov function $V = x^T Px$ such that for some positive definite matrix P, we have $\frac{dV}{dt}$ is negative definite), to establish the optimality condition for any stabilizing system. Let's consider the following theorem.

Theorem 4.2: For some Lyapunov candidate function $V = x^T Px$, if the stabilizing state feedback controller $u^* = -Kx$ applied for the LTI system (4.95) is such that $\min_u \left(\frac{dV}{dt} + x^T Qx + u^T Ru + 2x^T Nu \right) = 0$, then the controller is optimal.

Proof of Theorem 4.2: See Appendix C – section C-6

The design of the optimal control problem can be solved by finding the appropriate Lyapunov function $V(x) = x^T Px$. The Lyapunov candidate matrix P can be obtained by minimizing the functional

$$\frac{dV}{dt} + x^T Qx + u^T Ru + 2x^T Nu \tag{4.132}$$

We apply the necessary condition of unconstrained minimization, discussed early in this chapter, to the equation (4.74). We will have

$$\frac{\partial}{\partial u}\left(\frac{dV}{dt} + x^T Qx + u^T Ru + 2x^T Nu \right)\Bigg|_{u=u^*} = 0 \tag{4.133}$$

Noting $V(x) = x^T Px$, we have

$$\begin{aligned}
\frac{\partial}{\partial u}\left(\frac{dV}{dt} + x^T Qx + u^T Ru + 2x^T Nu \right) &= \frac{\partial}{\partial u}\left(2x^T P\dot{x} + x^T Qx + u^T Ru + 2x^T Nu \right) \\
&= \frac{\partial}{\partial u}\left(2x^T PAx + 2x^T PBu + x^T Qx + u^T Ru + 2x^T Nu \right) \\
&= 2\left(B^T Px + Ru + N^T x \right) = 0
\end{aligned} \tag{4.134}$$

Hence, from (4.134), we find the optimal control law is given by (4.122), *i.e.*,

$$u^* = -Kx \tag{4.135}$$

with

$$K = R^{-1}\left(PB + N \right)^T \tag{4.136}$$

the static state feedback gain.

We check now the sufficiency condition of the unconstrained condition

$$\frac{\partial^2}{\partial u^2}\left(\frac{dV}{dt}+x^TQx+u^TRu+2x^TNu\right)=\frac{\partial^2}{\partial u^2}\left(2x^TPAx+2x^TPBu+x^TQx+u^TRu+2x^TNu\right)$$

$$=2\frac{\partial}{\partial u}\left(B^TPx+Ru+N^Tx\right)$$

$$=2R>0 \tag{4.137}$$

since R is symmetric positive definite. Hence the second order sufficiency condition is satisfied.

Now, we calculate the matrix P. From (4.95), (4.135) and (4.136), the optimal closed loop system has the form

$$\dot{x}=A-BR^{-1}\left(B^TP+N^T\right)x \tag{4.138}$$

The optimal controller u^* satisfies the condition (4.133). So,

$$2x^TPAx+2x^TPBu^*+x^TQx+u^{*T}Ru^*+2x^TNu^*=0 \tag{4.139}$$

We substitute the expression for u^*, given by (4.138) into (4.139) to write

$$x^T\left(A^TP+PA\right)x-2x^TPBR^{-1}\left(PB+N\right)^Tx+x^TQx+x^T\left(PB+N\right)R^{-1}\left(PB+N\right)^Tx-$$
$$2x^TNR^{-1}\left(PB+N\right)^Tx=0 \tag{4.140}$$

which leads to

$$x^T\left(A^TP+PA\right)x-2x^T\left(PB+N\right)R^{-1}\left(PB+N\right)^Tx+x^TQx+$$
$$x^T\left(PB+N\right)R^{-1}\left(PB+N\right)^Tx=0 \tag{4.141}$$

Factoring out x and x^T yields

$$x^T\left(A^TP+PA+Q-\left(PB+N\right)R^{-1}\left(PB+N\right)^T\right)x=0 \tag{4.142}$$

The above equation should be true for every x. In other words, we have

$$A^T P + PA + Q - (PB + N) R^{-1} (PB + N)^T = 0 \tag{4.143}$$

The above equation is exactly the same as (4.111) referring to the general ARE. In conclusion, the resolution of the optimal control problem, minimizing the performance index (4.96) subject to the dynamics (4.95), leads simply to the computation of the ARE (4.143).

4.4.4.3 ARE – Main results

Given A, B, N, Q and R matrices of dimensions $(n \times n)$, $(n \times m)$, $(n \times m)$, $(n \times n)$ and $(m \times m)$ respectively. Q and R are symmetric non-negative definite and symmetric positive definite. Consider the following ARE

$$A^T P + PA + Q - (PB + N) R^{-1} \left(B^T P + N^T \right) = 0 \tag{4.144}$$

which is equivalent to the equation

$$\left(A - BR^{-1} N^T \right)^T P + P \left(A - BR^{-1} N^T \right) + Q - NR^{-1} N^T - PBR^{-1} B^T P = 0 \tag{4.145}$$

in the matrix P of dimensions $(n \times n)$.

Definition 4.1: An unforced dynamical LTI system $\dot{x} = Ax$ is said to be stable if all eigenvalues of A are in the open left half plane, that is, $\mathrm{Re}\left[\lambda (A) \right] < 0$. A matrix A with such a property is said to be asymptotically stable or Hurwitz [4-47].

Definition 4.2: The LTI system $\dot{x} = Ax + Bu$ is stabilizable if all unstable modes are controllable (*i.e.*, all uncontrollable modes are stable) [47-48].

Theorem 4.3: The dynamical system $\dot{x} = Ax + Bu$, or (A, B) is said to be stabilizable if there exists a state feedback $u = Kx$, such that the system is stable (*i.e.*, $A + BK$ is stable) [47-48].

Definition 4.3: The LTI system

$$\dot{x} = Ax + Bu \tag{4.146}$$

$$y = Cx + Du \tag{4.147}$$

is detectable if all unstable modes are observable, *i.e.*, all unobservable modes are stable [47-48].

Theorem 4.4: The dynamical system (4.146) and (4.147) or the pair (A,C) is said to be detectable if there exists a matrix L such that $A+LC$ is Hurwitz, *i.e.*, asymptotically stable [47-48].

Definition 4.4: The observability map of the pair (A,C) is given by the function [47-49]: $¸_0 \ : \mathbb{R}^n \rightarrow L_2([0,t_1],\mathbb{R}^p)$ *s.t.* $x_0 \rightarrow C \cdot e^{At} \cdot x_0 \quad \forall t \in [0,t_1]$.

Theorem 4.5: The following statements are equivalent [47-49].

1. The pair (A,C) is observable on $[0,t_1]$.
2. $Ker(¸_0) = \{0\}$.

Corollary 4.1: The Sylvester operator $¸(X) = AX + XB$ is singular if A and B share common eigenvalues.

Lemma 4.1: Consider the Sylvester equation $AX + XB = C$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$ and $C \in \mathbb{R}^{n\times m}$ are given matrices. There exists a unique solution $X \in \mathbb{R}^{n\times m}$ if, and only if, $a_i(A) + a_j(B) \neq 0 \quad ,\forall i = 1,2,\cdots,n$ and $j = 1,2,\cdots,m$ [47].

Theorem 4.6: Assume the matrix $Q - NR^{-1}N^T$ is symmetric, non-negative definite, i.e., $Q - NR^{-1}N^T \geq 0$. If (A,B) is stabilizable and $(A - BR^{-1}N^T, Q - NR^{-1}N^T)$ is detectable, then P solution of (4.144) is unique and symmetric and $A - BR^{-1}(B^T P + N^T)$ is Hurwitz [46-47].

Proof of Theorem 4.6: See Appendix C – section C-7

Now let we prove that $\bar{A} = A - BR^{-1}N^T - BR^{-1}B^T P$ is asymptotically stable.

Proposition 4.1: A_c is Hurwitz if one of the following is true

1. $\bar{Q} = Q - NR^{-1}N^T > 0$
2. $\bar{Q} \geq 0$ and $(\bar{A},\bar{Q})$ is observable.

In fact, consider the ARE (4.143). Suppose x an eigenvector of the matrix A_c, *i.e.*, $A_c x = ª x$ for ª the associated eigenvalue. Pre-multiply and post-multiply (4.143) by x^T and x respectively and use $R^{-1} = \bar{R}\bar{R}$, *i.e.*,

$$x^T P^T A_c x + x^T A_c^T P x + x^T \bar{Q} x + x^T P B \bar{R}^T \bar{R} B^T P x = 0 \tag{4.148}$$

which leads to

$$2 ª x^T P x = -x^T \bar{Q} x - \left\| \bar{R} B^T P x \right\|^2 \tag{4.149}$$

1[st] case: Suppose $\bar{Q} > 0$, then $-x^T \bar{Q} x - \left\| \bar{R} B^T P x \right\|^2 < 0$. Thus, $2 ª x^T P x < 0$ and $P > 0$. We conclude $\mathrm{Re}(ª) < 0$, that is, A_c is Hurwitz.

2[nd] case: Suppose $\bar{Q} \geq 0$ and $\left(\bar{A}, \bar{Q}\right)$ is observable. Assume $\mathrm{Re}(ª) = 0$ or $ª = jw$. We have from (4.252)

$$2\,\mathrm{Re}(ª)\, x^T P x = 0 = -x^T \bar{Q} x - \left\| \bar{R} B^T P x \right\|^2 \tag{4.150}$$

Then, $x^T \bar{Q} x = 0$ and $\left\| \bar{R} B^T P x \right\| = 0$, *i.e.*, $\bar{Q} x = 0$ and $B^T P x = 0$. We deduce

$$A_c x = \bar{A} x - B R^{-1} B^T P x = \bar{A} x = ª x = jwx \tag{4.151}$$

Noting $e^{\bar{A}t} = \sum_{i \geq 0} \frac{\bar{A}^i t^i}{i!}$, we write

$$e^{\bar{A}t} x = \sum_{i \geq 0} t^i \frac{\bar{A}^i}{i!} x = \sum_{i \geq 0} \frac{t^i}{i!} (jw)^i x$$

$$= \sum_{i \geq 0} \frac{(tjw)^i}{i!} x = e^{jwt} x \tag{4.152}$$

Thus,

$$\bar{Q} e^{\bar{A}t} x = \bar{Q} e^{jwt} x = e^{jwt} \bar{Q} x = 0 \tag{4.153}$$

which is a contradiction to $\left(\bar{A},\bar{Q}\right)$ observable.

Now, we prove that P is solution of the ARE (4.143). In fact, suppose that there exists two solutions P_1 and P_2 such that $A_{c_1}=\bar{A}-BR^{-1}B^TP_1$ and $A_{c_2}=\bar{A}-BR^{-1}B^TP_2$ are stable. From (4.235), we write

$$P_1^TA_{c_1}+A_{c_1}^TP_1+\bar{Q}+P_1^TBR^{-1}B^TP_1=0 \tag{4.154}$$

and

$$P_2^TA_{c_2}+A_{c_2}^TP_2+\bar{Q}+P_2^TBR^{-1}B^TP_2=0 \tag{4.155}$$

Subtract $P_2^TA_{c_1}$ from the ARE (4.154)

$$\left(P_1^T-P_2^T\right)A_{c_1}+A_{c_1}^TP_1+\bar{Q}=-P_2^TA_{c_1}-P_1^TBR^{-1}B^TP_1=0 \tag{4.156}$$

Subtract $A_{c_2}^TP_1$ from the ARE (4.155)

$$P_2^TA_{c_2}+A_{c_2}^T\left(P_2-P_1\right)+\bar{Q}=-A_{c_2}^TP_1-P_2^TBR^{-1}B^TP_2 \tag{4.157}$$

Subtracting (4.156) and (4.157) leads to

$$\left(P_1^T-P_2^T\right)A_{c_1}+A_{c_2}^T\left(P_2-P_1\right)=\left(A_{c_2}^T-A_{c_1}^T\right)P_1+P_2^T\left(A_{c_2}-A_{c_1}\right)+P_2^TBR^{-1}B^TP_2-$$
$$P_1^TBR^{-1}B^TP_1 \tag{4.158}$$

Note $A_{c_1}=\bar{A}-BR^{-1}B^TP_1$ and $A_{c_2}=\bar{A}-BR^{-1}B^TP_2$. (4.158) is written

$$\begin{aligned}\left(P_1^T-P_2^T\right)A_{c_1}+A_{c_2}^T\left(P_2-P_1\right)&=\left(\bar{A}-BR^{-1}B^TP_2-\bar{A}+BR^{-1}B^TP_1\right)^TP_1\\&\quad+P_2^TBR^{-1}B^TP_2-P_1^TBR^{-1}B^TP_1\\&\quad+P_2^T\left(\bar{A}-BR^{-1}B^TP_2-\bar{A}+BR^{-1}B^TP\right)\\&=-P_2^TBR^{-1}B^TP_1+P_1^TBR^{-1}B^TP_1-P_2^TBR^{-1}B^TP_2\\&\quad+P_2^TBR^{-1}B^TP_2-P_1^TBR^{-1}B^TP_1\end{aligned} \tag{4.159}$$

i.e.,

$$(P_1 - P_2)^T A_{c_1} + A_{c_2}^T (P_1 - P_2) = 0 \qquad (4.160)$$

Since A_{c_1} and A_{c_2} are both asymptotically stable, then $\forall i, \operatorname{Re}\left(\lambda_i\left(A_{c_1}\right)\right) < 0$ and $\operatorname{Re}\left(\lambda_i\left(A_{c_2}\right)\right) < 0$. Thus, the unique solution for (4.160) is

$$P_1 - P_2 = 0 \text{ , i.e., } P_1 = P_2 \qquad (4.161)$$

Finally, note that the solution of the ARE is symmetric. In fact, P is a solution of (4.143) will lead to P^T solution of (4.143). As P is unique, then $P^T = P$,*i.e.*, P symmetric.

4.4.4.4 Illustration example of an LQR problem

In this section, we illustrate the LQR problem of 2-DOF inverted penduli coupled with a spring plant as shown in Figure 4.1 [51].

The variables are: θ_i angular displacement of pendulum $i\,(i=1,2)$, τ_i torque input generated by the actuator for pendulum $i\,(i=1,2)$, F spring force, $\tilde{l}$ spring length, α slope of the spring to the earth, l_i length of pendulum $i\,(i=1,2)$, m_i mass of pendulum $i\,(i=1,2)$, L distance between the two penduli , and k spring constant.

The equations of motion of the two inverted pendulum system are [51]

$$m_1 \frac{l_1^2}{2} \ddot{\theta}_1 = \tau_1 + m_1 g \frac{l_1}{2} \sin(\theta_1) + l_1 F \cos(\theta_1 - \alpha) \qquad (4.162)$$

and

$$m_2 \frac{l_2^2}{2} \ddot{\theta}_2 = \tau_2 + m_2 g \frac{l_2}{2} \sin(\theta_2) - l_2 F \cos(\theta_2 - \alpha) \qquad (4.163)$$

where g is the constant of gravity,

$$F = k\left(\tilde{l} - \left[L^2 + (l_2 - l_1)^2\right]^{1/2}\right) \qquad (4.164)$$

$$\tilde{l} = \left[\left(L + l_2 \sin\theta_2 - l_1 \sin\theta_1 \right)^2 + \left(l_2 \cos\theta_2 - l_1 \cos\theta_1 \right)^2 \right]^{1/2} \tag{4.165}$$

and

$$\phi = \tan^{-1}\left(\frac{l_1 \cos\theta_1 - l_2 \sin\theta_2}{L + l_2 \sin\theta_2 - l_1 \sin\theta_1} \right) \tag{4.166}$$

with the initial conditions

$$F = 0 \text{ when } \theta_1 = \theta_2 = 0 \tag{4.167}$$

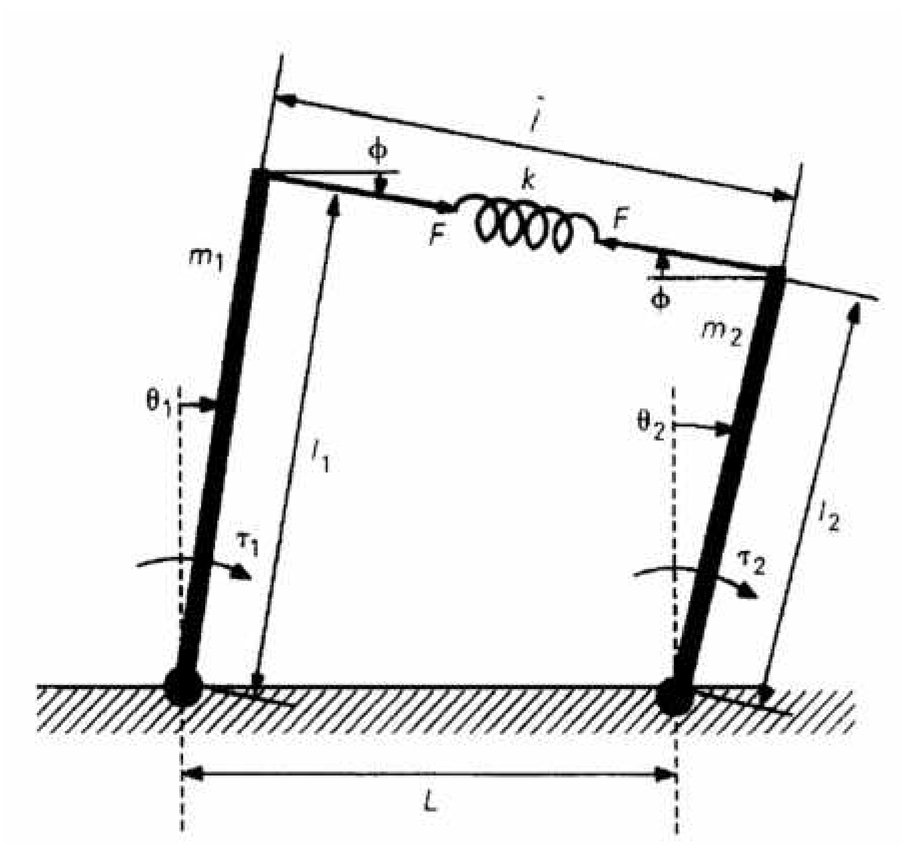

Figure 4.1 Two inverted pendulum coupled by a spring system [51]

This implies

$$\left(\text{—}_1, \dot{\text{—}}_1, \text{—}_2, \dot{\text{—}}_2\right)^T = 0 \tag{4.168}$$

is an equilibrium state of the system if $\text{²}_1 = \text{²}_2$. The system (4.162) and (4.163) is written in a matrix form as [51]

$$\frac{d}{dt}\begin{bmatrix} \text{—}_1 \\ \dot{\text{—}}_1 \\ \text{—}_2 \\ \dot{\text{—}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{g}{l_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l_2} & 0 \end{bmatrix}\begin{bmatrix} \text{—}_1 \\ \dot{\text{—}}_1 \\ \text{—}_2 \\ \dot{\text{—}}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{2}{m_1 l_1^2} & 0 \\ 0 & 0 \\ 0 & \frac{2}{m_2 l_2^2} \end{bmatrix}\begin{bmatrix} \text{²}_1 \\ \text{²}_2 \end{bmatrix}$$
$$+\begin{bmatrix} 0 \\ \frac{g}{l_1}\left(\sin\text{—}_1 - \text{—}_1\right) + \frac{2F}{m_1 l_1}\cos\left(\text{—}_1 - \text{¤}\right) \\ 0 \\ \frac{g}{l_2}\left(\sin\text{—}_2 - \text{—}_2\right) - \frac{2F}{m_2 l_2}\cos\left(\text{—}_2 - \text{¤}\right) \end{bmatrix} \tag{4.169}$$

We note

$$x = \begin{bmatrix} \text{—}_1 \\ \dot{\text{—}}_1 \\ \text{—}_2 \\ \dot{\text{—}}_2 \end{bmatrix} \tag{4.170}$$

the state vector,

$$u = \begin{bmatrix} \text{²}_1 \\ \text{²}_2 \end{bmatrix} \tag{4.171}$$

the control vector,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{g}{l_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l_2} & 0 \end{bmatrix} \tag{4.172}$$

a constant matrix,

$$B = \begin{bmatrix} 0 & 0 \\ \frac{2}{m_1 l_1^2} & 0 \\ 0 & 0 \\ 0 & \frac{2}{m_2 l_2^2} \end{bmatrix} \tag{4.173}$$

a constant matrix, and

$$f_c = \begin{bmatrix} 0 \\ \frac{g}{l_1}\left(\sin\text{¯}_1 - \text{¯}_1\right) + \frac{2F}{m_1 l_1}\cos\left(\text{¯}_1 - \text{¤}\right) \\ 0 \\ \frac{g}{l_2}\left(\sin\text{¯}_2 - \text{¯}_2\right) - \frac{2F}{m_2 l_2}\cos\left(\text{¯}_2 - \text{¤}\right) \end{bmatrix} \tag{4.174}$$

the perturbation vector which is nonlinear in x. Hence, the dynamics (4.219) can be written

$$\dot{x} = Ax + Bu + f_c\left(x\right) \tag{4.175}$$

To linarize the system (4.225), we assume that the nonlinear term $f_c = 0$, then we obtain the dynamics

$$\dot{x} = Ax + Bu \tag{4.176}$$

The performance index associated with this dynamics is defined by

$$J = \frac{1}{2}\int_0^\infty \left(x^T Q x + u^T R u \right) dt \tag{4.177}$$

where Q and R are the weighting matrices given by

$$Q = \begin{bmatrix} m_1 g & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & 10 m_2 g & 0 \\ 0 & 0 & 0 & 10 m_2 \end{bmatrix} \tag{4.178}$$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.179}$$

Then, the optimal control law is given by

$$u^* = -R^{-1} B^T P^T x \tag{4.180}$$

where P is solution of the ARE

$$A^T P + PA + Q - PBR^{-1} B^T P = 0 \tag{4.181}$$

For the numerical application, consider the constant values

$$\begin{cases} l_1 = 1m \\ l_2 = 0.8m \\ m_1 = 1kg \\ m_2 = 0.8kg \\ g = 9.8m/s^2 \\ k = 0.02N/m \end{cases} \tag{4.182}$$

Using MATLAB, we obtain the solution of ARE (4.181) as follows

$$P = \begin{bmatrix} 19.93 & 5.36 & 0 & 0 \\ 5.36 & 1.71 & 0 & 0 \\ 0 & 0 & 35.69 & 3.21 \\ 0 & 0 & 3.21 & 0.97 \end{bmatrix} \tag{4.183}$$

Hence, using (4.284), we obtain the optimal control law $u^* = -Kx$, where

$$K = \begin{bmatrix} 10.72 & 3.42 & 0 & 0 \\ 0 & 0 & 12.54 & 3.80 \end{bmatrix} \tag{4.184}$$

Using SIMULINK, we simulate the linearized system (4.280) using the optimal control law (4.284) for different initial conditions. The time evolution of the angular positions ¯$_1$, ¯$_2$ and the actuator torques ²$_1$, ²$_2$ for an initial condition of ¯$_1 = \frac{®}{5}$ and ¯$_2 = \frac{®}{6}$ are shown in Figures 4.2 to 4.5.

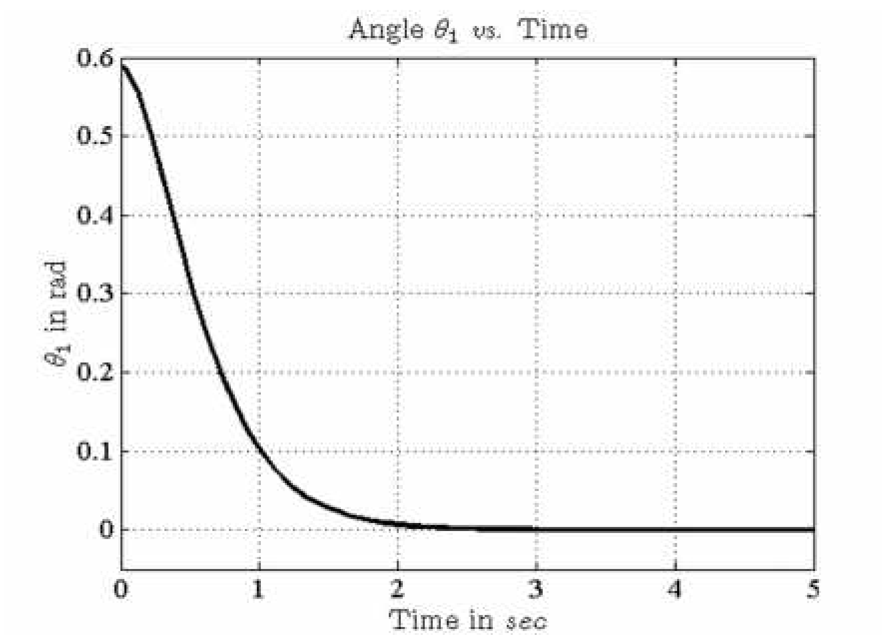

Figure 4.2 ¯$_1$ evolution *vs.* time of the two inverted pendulum system

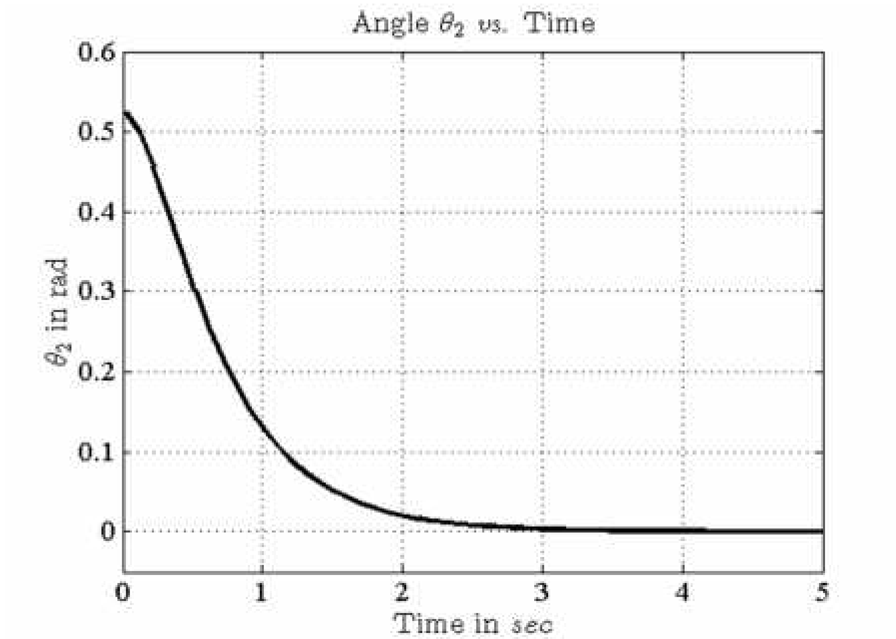

Figure 4.3 ¯$_2$ evolution *vs.* time of the two inverted pendulum system

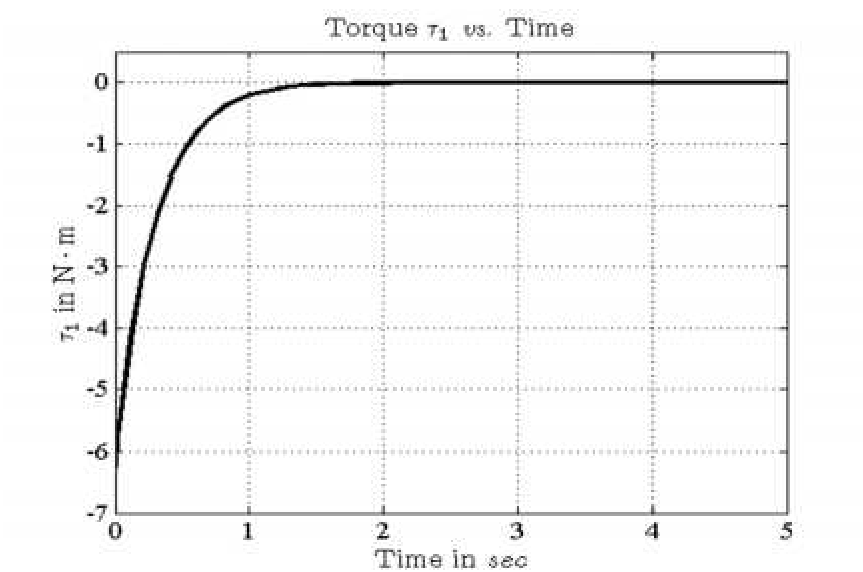

Figure 4.4 $^2{}_1$ evolution *vs.* time of the two inverted pendulum system

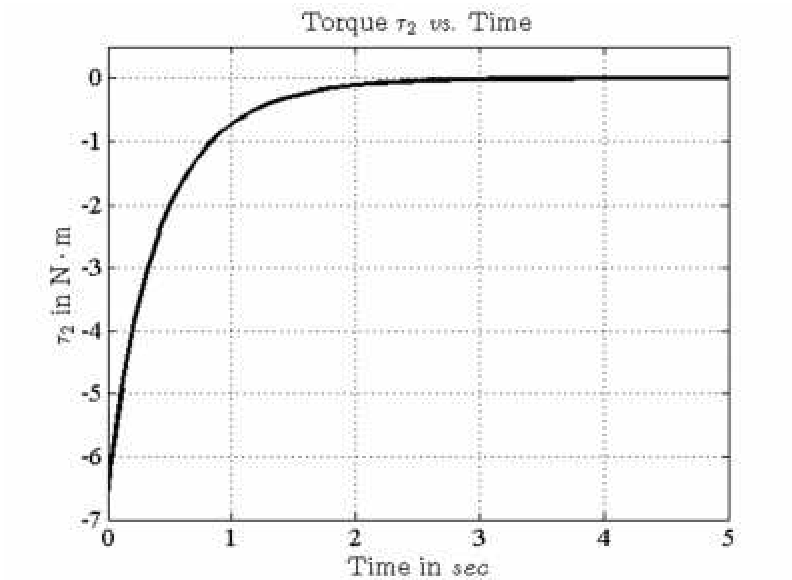

Figure 4.5 τ_2 evolution *vs.* time of the two inverted pendulum system

The simulation results for the angular positions θ_1 and θ_2, show that the LQR stabilize the system at the equilibrium position. The input signals (*i.e.*, motor torques τ_1 and τ_2) show that both motors present a normal behaviour and there is no important overshoot that will cause saturation issues.

4.5 Conclusion

In this chapter, we introduced the problem of optimization and some applications in engineering as well as its classifications based on several criteria. We presented the problem of optimization and some applications in engineering as well as its classification based on several aspects. Then, in section 4.2, we presented the problem of optimization with no constraints in which we treated first the problem of multivariable function optimization with no constraints, then the problem of functional minimization with no constraints and finally we presented the calculus of variation problem statement and their solutions. In section 4.3, we treated the optimization problem with equality constraints. We began with the general problem of multivariable function optimization with constraints, then we solved the particular problem of minimization with equality constraints, and finally, we treated the method of Lagrange multipliers under three different cases. In section 4.4, we presented the optimal control theory. We began with the formulation of general problem, then we treated the HJE, then we presented the infinite horizon problem and the LQR controller. We presented the design method and then we studied the stability of such controller and how this leads to the main results of ARE. And finally, we illustrated the LQR problem through an example. In section 4.5, we conclude this chapter.

5 Optimal control of polynomial systems using Kronecker product

5.1 Introduction

The objective of this chapter is to compute an optimal control law for polynomial systems using the KP formulation. In section 5.2, we will state the problem and we show how the problem of finding the optimal solution is reduced in solving what we call the State Dependent Ricatti equation (SDR). In section 5.3, the problem of solving the SDR is transformed into solving uncoupled linear equations in the gain matrices using the KP algebra. The calculation of these gain matrices is presented in section 5.4. It's done by the cancellation of the coefficients of VPS terms and the resolution of linear equations. In section 5.5, we apply the proposed method to three scalar examples.

5.2 Problem statement

Consider the nonlinear dynamics given by

$$\dot{x}(t) = F(x) + G(x) \cdot u(t) = F(x) + \sum_{k=1}^{m} G_k(x) \cdot u_k(t) \tag{5.1}$$

where $t \in \mathbb{R}$ is the continuous time, $x(t) \in \mathbb{R}^n$ the state vector and $u(t) = \left[u_1(t) \quad .... \quad u_m(t) \right]^T \in \mathbb{R}^m$ the input vector. $F(.), G_k(.)$ for $k = 1,...,m$ are analytic vector fields from $\mathbb{R}^n$ into $\mathbb{R}^n$ given by the following polynomials

$$F(x) = \sum_{j=1}^{l} F_j \cdot x^{|j|} \tag{5.2}$$

$$G_k(x) = \sum_{j=0}^{g} G_{kj} \cdot x^{|j|} \quad \text{for } k = 1, \dots, m \tag{5.3}$$

$$G(x) = \sum_{j=0}^{g} G_j \left(I_m \otimes x^{|j|} \right) \tag{5.4}$$

with $G_j = \left[G_{1j} \vdots \quad \cdots \quad \vdots G_{mj} \right] \in \mathbb{R}^n \times \mathbb{R}^{mn^j}$. In the following, we treat the generalized cost function of the form

$$J = \frac{1}{2} \int_0^\infty \left(s^T(t) Q s(t) + u^T(t) R u(t) + 2 s^T(t) N u(t) \right) dt \tag{5.5}$$

where $s(t)$ is the output vector of $\mathbb{R}^q$

$$s(x) = \sum_{j=1}^{k} H_j x^{|j|} \tag{5.6}$$

R is a positive definite matrix of $\mathbb{R}^{m \times m}$, Q a non-negative matrix of $\mathbb{R}^{q \times q}$ and N a matrix of $\mathbb{R}^{p \times m}$ .In addition, we assume that the matrix $Q - NR^{-1}N^T$ is non-negative definite. This work is an extension of the optimal control problem based on the KP algebra introduced by Rotella and Tunguy [54.]

Following Boudarel *et al.* [54] and using (4.133), introduced in chapter 4, we denote by $V(x(t))$ the optimal cost with an initial condition x at t

$$V(x) = \frac{1}{2} \int_t^\infty \left(s^T(x(2)) Q s(x(2)) + u^{*T}(2) R u^*(2) + 2 s^T(x(2)) N u^*(2) \right) d2 \tag{5.7}$$

where $u^* = \arg(\min J)$ is the optimal control. Then, we write the HJE [3-53]

$$\frac{\partial V}{\partial t}(x) = \min_u \left[\frac{1}{2} \left(s^T(x) Q s(x) + u^T R u + 2 s^T(x) N u \right) + \left(\frac{\partial V}{\partial x} \right)^T \left(F(x) + G(x) u \right) \right] \tag{5.8}$$

That is, referring to (4.151) and (4.152)

$$\frac{\partial V}{\partial t}(x) = -\left[\frac{1}{2}\left(s^T(x)Qs(x) + u^{*T}Ru^* + 2s^T(x)Nu^*\right) + \left(\frac{\partial V}{\partial x}\right)^T \left(F(x) + G(x)u\right)\right] \quad (5.9)$$

Considering the stationary infinite time and non-constraint input case [3], we have $\frac{\partial V}{\partial t} = 0$. Thus, from (5.8), we obtain [3-43]

$$u^* = -R^{-1}\left(N^T s(x) + G^T(x)\frac{\partial V}{\partial x}\right) \quad (5.10)$$

substituting (5.10) in (5.9) leads to

$$s^T(x)Qs(x) - s^T(x)NR^{-1}N^T s(x) - \frac{\partial V^T}{\partial x}G(x)R^{-1}G^T(x)\frac{\partial V}{\partial x} + \frac{\partial V^T}{\partial x}F(x) + F^T(x)\frac{\partial V}{\partial x}$$

$$-\frac{\partial V^T}{\partial x}G(x)R^{-1}N^T s(x) - s^T(x)NR^{-1}G^T(x)\frac{\partial V}{\partial x} = 0 \quad (5.11)$$

or equivalently,

$$\left[F(x) - G(x)R^{-1}N^T s(x)\right]^T\left(\frac{\partial V}{\partial x}\right) + \left(\frac{\partial V}{\partial x}\right)^T\left[F(x) - G(x)R^{-1}N^T s(x)\right]$$

$$-\left(\frac{\partial V}{\partial x}\right)^T\left[G(x)R^{-1}G^T(x)\right]\left(\frac{\partial V}{\partial x}\right) + s^T(x)\left[Q - NR^{-1}N^T\right]s(x) = 0 \quad (5.12)$$

The equations (5.10) and (5.12) determine the optimal control law u^*. The term $\frac{\partial V}{\partial x}$ is calculated first through the equation (5.12), and after that, through the equation (5.10), we can calculate u^*.

5.3 Equation of approximation

Merriam [55] and Lukes [56] have proposed the determination of an analytic expression for $\frac{\partial V}{\partial x}$, and then for u^*. They assume that $\frac{\partial V}{\partial x}$ can be written in a polynomial form as

$$\frac{\partial V}{\partial x} = \sum_{j=1}^{\bar{p}} P_j x^{|j|} \tag{5.13}$$

where P_j, $j \geq 1$, are constant matrices of $\mathbb{R}^{n \times n^j}$. Using the KP proprieties, *vec* and *mat* notations, introduced in chapter 3, we transform (5.12) by substituting (5.2), (5.4), (5.14) and (5.6), to obtain

$$\begin{aligned}
&\left(\left(\sum_{i=1}^{f} F_i x^{|i|}\right) - \left(\sum_{i=0}^{g} G_i\left(I_m \otimes x^{|i|}\right)\right) R^{-1} N^T \left(\sum_{i=1}^{h} H_i x^{|i|}\right)\right)^T \left(\sum_{i=1}^{\bar{p}} P_i x^{|i|}\right) \\
&+\left(\sum_{i=1}^{\bar{p}} P_i x^{|i|}\right)^T \left(\left(\sum_{i=1}^{f} F_i x^{|i|}\right) - \left(\sum_{i=0}^{g} G_i\left(I_m \otimes x^{|i|}\right)\right) R^{-1} N^T \left(\sum_{i=1}^{h} H_i x^{|i|}\right)\right) \\
&-\left(\sum_{i=1}^{\bar{p}} P_i x^{|i|}\right)^T \left(\sum_{i=0}^{g} G_i\left(I_m \otimes x^{|i|}\right)\right) R^{-1} \left(\sum_{i=0}^{g} G_i\left(I_m \otimes x^{|i|}\right)\right)^T \left(\sum_{i=1}^{\bar{p}} P_i x^{|i|}\right) \\
&+\left(\sum_{i=1}^{h} H_i x^{|i|}\right)^T \left(Q - NR^{-1}N^T\right)\left(\sum_{i=1}^{h} H_i x^{|i|}\right) = 0
\end{aligned} \tag{5.14}$$

This is equivalent to

$$\begin{aligned}
&\sum_{i=1}^{f}\sum_{j=1}^{\bar{p}} x^{|i|T} F_i^T P_j x^{|i|} + \sum_{i=1}^{\bar{p}}\sum_{j=1}^{f} x^{|i|T} P_i^T F_j x^{|j|} + \sum_{i=1}^{h}\sum_{j=1}^{h} x^{|i|T} H_i^T \left(Q - NR^{-1}N^T\right) H_j x^{|j|} \\
&-\sum_{i=1}^{h}\sum_{j=0}^{g}\sum_{k=1}^{\bar{p}} x^{|i|T} H_i^T NR^{-1}\left(I_m \otimes x^{|j|T}\right) G_j^T P_k x^{|k|} - \sum_{i=1}^{\bar{p}}\sum_{j=0}^{g}\sum_{k=1}^{h} x^{|i|T} P_i^T G_j \left(I_m \otimes x^{|j|}\right) R^{-1} N^T H_k x^{|k|} \\
&-\sum_{i=1}^{\bar{p}}\sum_{j=0}^{g}\sum_{c=1}^{g}\sum_{d=0}^{\bar{p}} x^{|i|T} P_i^T G_j \left(I_m \otimes x^{|j|}\right) R^{-1} \left(I_m \otimes x^{|c|T}\right) G_c^T P_d x^{|d|} = 0
\end{aligned} \tag{5.15}$$

By using Theorems 3.14, 3.17, Lemmas 3.2 and 3.3, introduced in chapter 3, (5.15) can be transformed into

$$\sum_{i=1}^{f}\sum_{j=1}^{\bar{p}} vec^T\left(F_i^T P_j\right)x^{|i+j|} + \sum_{i=1}^{\bar{p}}\sum_{j=1}^{f} vec^T\left(P_i^T F_j\right)x^{|i+j|} + \sum_{i=1}^{h}\sum_{j=1}^{h} vec^T\left(H_i^T\left(Q-NR^{-1}N^T\right)H_j\right)x^{|i+j|}$$
$$-\sum_{i=1}^{h}\sum_{j=0}^{g}\sum_{k=1}^{\bar{p}} x^{|i|T}H_i^T NR^{-1} mat^T_{n^{j+k}\times m}\left[vec\left(P_i^T F_j\right)\right]x^{|i+j|} - \sum_{i=1}^{\bar{p}}\sum_{j=0}^{g}\sum_{k=1}^{h} x^{|k|T}H_k^T NR^{-1}\left(I_m \otimes x^{|j|T}\right)G_j^T P_i x^{|i|}$$
$$-\sum_{i=1}^{\bar{p}}\sum_{j=0}^{g}\sum_{c=1}^{g}\sum_{d=0}^{\bar{p}}\left[\left(I_m \otimes x^{|j|T}\right)G_j^T P_i x^{|i|}\right]^T R^{-1} mat^T_{n^{c+d}\times m}\left[vec\left(P_d^T G_c\right)\right]x^{|c+d|} = 0 \qquad (5.16)$$

This is equivalent to

$$\sum_{i=1}^{f}\sum_{j=1}^{\bar{p}} vec^T\left(F_i^T P_j\right)x^{|i+j|} + \sum_{i=1}^{f}\sum_{j=1}^{\bar{p}} vec^T\left(P_i^T F_j\right)x^{|i+j|} + \sum_{i=1}^{h}\sum_{j=1}^{h} vec^T\left(H_i^T\left(Q-NR^{-1}N^T\right)H_j\right)x^{|i+j|}$$
$$-\sum_{i=1}^{h}\sum_{j=0}^{g}\sum_{k=1}^{\bar{p}} x^{|i|T}H_i^T NR^{-1}V_{kj}^T x^{|i+j|} - \sum_{i=1}^{\bar{p}}\sum_{j=0}^{g}\sum_{k=1}^{h} x^{|k|T}H_k^T NR^{-1} mat^T_{n^{i+j}\times m}\left[vec\left(P_i^T G_j\right)\right]x^{|i+j|}$$
$$-\sum_{i=1}^{\bar{p}}\sum_{j=0}^{g}\sum_{c=1}^{g}\sum_{d=0}^{\bar{p}}\left[mat^T_{n^{i+j}\times m}\left[vec\left(P_i^T G_j\right)\right]x^{|i+j|}\right]^T R^{-1}V_{dc}^T x^{|c+d|} = 0 \qquad (5.17)$$

with $V_{kj} = mat_{n^{j+k}\times m}\left[vec\left(P_k^T G_j\right)\right] = \left[vec\left(P_k^T G_{1j}\right)\vdots \ \cdots \ \vdots vec\left(P_k^T G_{mj}\right)\right]$, notice that $V_{kj} = \left[vec\left(P_k^T G_{1j}\right)\vdots \ \cdots \ \vdots vec\left(P_k^T G_{mj}\right)\right]$ as noted in [3]. Hence, we write

$$\sum_{i=1}^{f}\sum_{j=1}^{\bar{p}} vec^T\left(F_i^T P_j\right)x^{|i+j|} + \sum_{i=1}^{\bar{p}}\sum_{j=1}^{f} P_i^T F_j x^{|i+j|} + \sum_{i=1}^{h}\sum_{j=1}^{h} vec^T\left(H_i^T\left(Q-NR^{-1}N^T\right)H_j\right)x^{|i+j|}$$
$$-\sum_{i=1}^{h}\sum_{j=0}^{g}\sum_{k=1}^{\bar{p}} vec^T\left(H_i^T NR^{-1}V_{kj}^T\right)x^{|i+j+k|} - \sum_{i=1}^{\bar{p}}\sum_{j=0}^{g}\sum_{k=1}^{h} H_k NR^{-1}V_{ij}^T x^{|i+j+k|}$$
$$-\sum_{i=1}^{\bar{p}}\sum_{j=0}^{g}\sum_{c=1}^{g}\sum_{d=0}^{\bar{p}} x^{|i+j|}V_{ij}R^{-1}V_{dc}^T x^{|c+d|} = 0 \qquad (5.18)$$

By applying again Theorem 3.14, we obtain

$$\sum_{i=1}^{f}\sum_{j=1}^{\bar{p}} vec^T\left(F_i^T P_j\right)x^{|i+j|}+\sum_{i=1}^{\bar{p}}\sum_{j=1}^{f} vec^T\left(P_i^T F_j\right)x^{|i+j|}+\sum_{i=1}^{h}\sum_{j=1}^{h} vec^T\left(H_i^T\left(Q-NR^{-1}N^T\right)H_j\right)x^{|i+j|}$$
$$-\sum_{i=1}^{h}\sum_{j=0}^{g}\sum_{k=1}^{\bar{p}} vec^T\left(H_i^T NR^{-1}V_{kj}^T\right)x^{|i+j+k|}-\sum_{i=1}^{\bar{p}}\sum_{j=0}^{g}\sum_{k=1}^{h} vec^T\left(V_{ij}R^{-1}N^T H_k\right)x^{|i+j+k|}$$
$$-\sum_{i=1}^{\bar{p}}\sum_{j=0}^{g}\sum_{c=1}^{g}\sum_{d=0}^{\bar{p}} vec^T\left(V_{ij}R^{-1}V_{dc}^T\right)x^{|i+j+c+d|}=0 \tag{5.19}$$

In order to compute the optimal control u^*, we have to find $\frac{\partial V}{\partial x}$, given in the polynomial form (5.13). So, we have to calculate the different terms P_i, $i\in\mathbb{N}^*$. These terms are obtained by cancelling the coefficients of $x^{|i+1|}$ in (5.19), which are the subject of the next section.

5.4 Determination of P_i

5.4.1 First order

The calculation of P_1 from (5.19) is given by the cancellation of the coefficients of $x^{|2|}$. We obtain

$$vec^T\left(P_1^T F_1\right)+vec^T\left(F_1^T P_1\right)+vec^T\left(H_1^T\left(Q-NR^{-1}N^T\right)H_1\right)$$
$$-vec^T\left(H_1^T NR^{-1}V_{10}\right)-vec^T\left(V_{10}R^{-1}N^T H_1\right)-vec^T\left(V_{10}R^{-1}V_{01}^T\right)=0 \tag{5.20}$$

Since the operator $vec(.)$ is linear on the matrices of the same dimensions, and $V_{10}=P_1^T G_0$, then the equation (5.20) will be

$$P_1^T F_1+F_1^T P_1+H_1^T\left(Q-NR^{-1}N^T\right)H_1-H_1^T NR^{-1}G_0^T P_1-P_1^T G_0 R^{-1}N^T H_1$$
$$-P_1^T G_0 R^{-1}G_0^T P_1=0 \tag{5.21}$$

P_1 is the gain matrix solution of the optimal control problem of the linearized system corresponding to (5.1). The equation (5.21) is the classical ARE introduced in chapter 4.

5.4.2 Second order

The calculation of P_2 from (5.19) is given by the cancellation of the coefficients of $x^{|3|}$. We obtain

$$\begin{aligned}
&vec^T\left(F_1^T P_2\right)+vec^T\left(F_2^T P_1\right)+vec^T\left(P_1^T F_2\right)+vec^T\left(P_2^T F_1\right)\\
&+vec^T\left(H_1^T\left(Q-NR^{-1}N^T\right)H_2\right)+vec^T\left(H_2^T\left(Q-NR^{-1}N^T\right)H_1\right)\\
&-vec^T\left(H_2^T NR^{-1}V_{10}^T\right)-vec^T\left(H_1^T NR^{-1}V_{11}^T\right)-vec^T\left(H_1^T NR^{-1}V_{20}^T\right)\\
&-vec^T\left(V_{10}R^{-1}N^T H_2\right)-vec^T\left(V_{11}R^{-1}N^T H_1\right)-vec^T\left(V_{20}R^{-1}N^T H_1\right)\\
&-vec^T\left(V_{11}R^{-1}V_{10}^T\right)-vec^T\left(V_{10}R^{-1}V_{11}^T\right)-vec^T\left(V_{20}R^{-1}V_{10}^T\right)-vec^T\left(V_{10}R^{-1}V_{20}^T\right)=0
\end{aligned} \quad (5.22)$$

We have $V_{20}=P_2^T G_0$ and using Theorem 3.9, given in chapter 3, (5.20) will be transformed into

$$\begin{aligned}
&\left(I_{n^3}+U_{n\times n^2}\right)vec^T\left(F_1^T P_2\right)+\left(I_{n^3}+U_{n\times n^2}\right)vec^T\left(P_1^T F_2\right)\\
&+\left(I_{n^3}+U_{n\times n^2}\right)vec^T\left(H_1^T\left(Q-NR^{-1}N^T\right)H_2\right)\\
&-\left(I_{n^3}+U_{n\times n^2}\right)vec^T\left(H_1^T NR^{-1}V_{11}^T\right)\\
&-\left(I_{n^3}+U_{n\times n^2}\right)vec^T\left(H_1^T NR^{-1}G_0^T P_2\right)-\left(I_{n^3}+U_{n\times n^2}\right)vec^T\left(V_{10}R^{-1}G_0^T P_2\right)\\
&-\left(I_{n^3}+U_{n\times n^2}\right)vec^T\left(V_{10}R^{-1}N^T H_2\right)-\left(I_{n^3}+U_{n\times n^2}\right)vec^T\left(V_{10}R^{-1}V_{11}^T\right)=0
\end{aligned} \quad (5.23)$$

Since $\left(I_{n^3}+U_{n\times n^2}\right)\neq 0$ (refer to Theorem 3..18 in chapter 3) and by the linearity of the $vec(.)$ operator (for the matrices inside has the same dimensions), the equation (5.23) becomes

$$P_1^T F_2 + F_1^T P_2 + H_1^T \left(Q - NR^{-1}N^T\right) H_2 - H_1^T NR^{-1} V_{11}^T - V_{10} R^{-1} N^T H_2 - V_{10} R^{-1} V_{11}^T$$

$$- H_1^T NR^{-1} G_0^T P_2 - V_{10} R^{-1} G_0^T P_2 = 0 \tag{5.24}$$

If we put the terms involving P_2 in one side and the other known terms in the other side, this will lead to

$$\left(H_1^T NR^{-1} G_0^T + V_{10} R^{-1} G_0^T - F_1^T\right) P_2 = H_1^T \left(Q - NR^{-1}N^T\right) H_2 + P_1^T F_2 - V_{10} R^{-1} N^T H_2$$

$$- H_1^T NR^{-1} V_{11}^T - V_{10} R^{-1} V_{11}^T \tag{5.25}$$

If we note

$$\mathscr{F}_2 = H_1^T NR^{-1} G_0^T + V_{10} R^{-1} G_0^T - F_1^T \tag{5.26}$$

$$\mathscr{H}_2 = H_1^T \left(Q - NR^{-1}N^T\right) H_2 + P_1^T F_2 - V_{10} R^{-1} N^T H_2 - H_1^T NR^{-1} V_{11}^T - V_{10} R^{-1} V_{11}^T \tag{5.27}$$

Then, the equation (5.25) will be

$$\mathscr{F}_2 P_2 = \mathscr{H}_2 \tag{5.28}$$

Hence, P_2 can be calculated as

$$P_2 = \mathscr{F}_2^{-1} \mathscr{H}_2 \tag{5.29}$$

In fact, note that P_1 is solution of the ARE (5.21). Using Theorem 4.3 in chapter 4 and noting $H_1^T \left(Q - NR^{-1}N^T\right) H_1$ is symmetric non-negative definite, assume $\left(F_1, G_0\right)$ is stabilizable and $\left(F_1 - G_0 R^{-1} N^T H_1 \,,\, H_1^T \left(Q - NR^{-1}N^T\right) H_1\right)$ is detectable, then $F_1 - G_0 R^{-1}\left(G_0^T P_1 + N^T H_1\right)$ is Hurwitz. Thus, $F_1 - G_0 R^{-1}\left(G_0^T P_1 + N^T H_1\right)$ is a regular matrix and its inverse exists.

5.4.3 General order

In the general case, to calculate P_p, $p \in \mathbb{N}^*$, we need to isolate the coefficients of $x^{|p+1|}$ in (5.19). By the cancellation of the coefficients of $x^{|p+1|}$ in (5.19), we obtain

$$\underbrace{\sum_{i=1}^{p}\sum_{j=1}^{p}}_{i+j=p+1} vec\left(P_i^T F_j\right) + \underbrace{\sum_{i=1}^{p}\sum_{j=1}^{p}}_{i+j=p+1} vec\left(F_i^T P_j\right) + \underbrace{\sum_{i=1}^{p}\sum_{j=1}^{p}}_{i+j=p+1} vec\left(H_a^T\left(Q - NR^{-1}N^T\right)H_b\right)$$

$$-\underbrace{\sum_{i=1}^{p}\sum_{j=0}^{p-1}\sum_{k=1}^{p}}_{i+j+k=p+1} vec\left(V_{ij}R^{-1}N^T H_k\right) - \underbrace{\sum_{i=1}^{p}\sum_{j=0}^{p-1}\sum_{k=1}^{p}}_{i+j+k=p+1} vec\left(H_k^T NR^{-1}V_{ij}^T\right)$$

$$-\underbrace{\sum_{i=1}^{p}\sum_{d=1}^{p}\sum_{j=0}^{p-1}\sum_{c=1}^{p-1}}_{i+j+d+c=p+1} vec\left(V_{ij}R^{-1}V_{dc}^T\right) = 0 \tag{5.30}$$

that is,

$$\sum_{i=1}^{p} vec\left(P_i^T F_{p-i+1}\right) + \sum_{i=1}^{p} vec\left(F_i^T P_{p-i+1}\right) + \sum_{i=1}^{p} vec\left(H_i^T\left(Q - NR^{-1}N^T\right)H_{p-i+1}\right)$$

$$-\underbrace{\sum_{i=1}^{p}\sum_{j=0}^{p-1}}_{1\le i+j\le p} vec\left(V_{ij}R^{-1}N^T H_{p+1-i-j}\right) - \underbrace{\sum_{i=1}^{p}\sum_{j=0}^{p-1}}_{1\le i+j\le p} vec\left(H_{p+1-i-j}^T NR^{-1}V_{ij}^T\right)$$

$$-\underbrace{\sum_{i,d=1}^{p}\sum_{j,c=0}^{p-1}}_{i+j+d+c=p+1} vec\left(V_{ij}R^{-1}V_{dc}^T\right) = 0 \tag{5.31}$$

By isolating all the terms in P_p from (5.31) and noting that $V_{p0} = P_p^T G_0$, we have

$$vec\left(P_p^T F_1\right)+\sum_{i=1}^{p-1} vec\left(P_i^T F_{p-i+1}\right)+vec\left(F_1^T P_p\right)+\sum_{i=2}^{p} vec\left(F_i^T P_{p-i+1}\right)$$

$$+\sum_{i=1}^{p} vec\left(H_i^T\left(Q-NR^{-1}N^T\right)H_{p-i+1}\right)-vec\left(P_p^T G_0 R^{-1}N^T H_1\right)-\underbrace{\sum_{i=1}^{p-1}\sum_{j=0}^{p-1}}_{1\le i+j\le p} vec\left(V_{ij}R^{-1}N^T H_{p+1-i-j}\right)$$

$$-vec\left(H_1^T NR^{-1}G_0^T P_p\right)-\underbrace{\sum_{i=1}^{p-1}\sum_{j=0}^{p-1}}_{1\le i+j\le p} vec\left(H_{p+1-i-j}^T NR^{-1}V_{ij}^T\right)-vec(P_p^T G_0 R^{-1}V_{10}^T)-vec(V_{10}R^{-1}G_0^T P_p)$$

$$-\underbrace{\sum_{i,d=1}^{p-1}\sum_{j,c=0}^{p-1}}_{1\le i+j\le p} vec\left(V_{ij}R^{-1}V_{dc}^T\right)=0 \tag{5.32}$$

If we group all the terms that of P_p in one side, and all the other terms in (P_{p-1} and below) in the other side, we obtain

$$vec\left(P_p^T F_1\right)+vec\left(F_1^T P_p\right)-vec\left(P_p^T G_0 R^{-1}N^T H_1\right)-vec\left(H_1^T NR^{-1}G_0^T P_p\right)$$

$$-vec(P_p^T G_0 R^{-1}V_{10}^T)-vec(V_{10}R^{-1}G_0^T P_p)=-\sum_{i=1}^{p-1} vec\left(P_i^T F_{p-i+1}\right)-\sum_{i=2}^{p} vec\left(F_i^T P_{p-i+1}\right)$$

$$-\sum_{i=1}^{p} vec\left(H_i^T\left(Q-NR^{-1}N^T\right)H_{p-i+1}\right)+\underbrace{\sum_{i=1}^{p-1}\sum_{j=0}^{p-1}}_{1\le i+j\le p} vec\left(V_{ij}R^{-1}N^T H_{p+1-i-j}\right)$$

$$+\underbrace{\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}}_{1\le i+j\le p} vec\left(H_{p+1-i-j}^T NR^{-1}V_{ij}^T\right)+\underbrace{\sum_{i,d=1}^{p-1}\sum_{j,c=0}^{p-1}}_{i+j+c+d=p+1} vec\left(V_{ij}R^{-1}V_{dc}^T\right) \tag{5.33}$$

We note

$$\mathscr{H}_p = -\sum_{i=1}^{p} vec\left(H_i^T\left(Q - NR^{-1}N^T\right)H_{p-i+1}\right) - \sum_{i=1}^{p-1} vec\left(P_i^T F_{p-i+1}\right) - \sum_{i=2}^{p} vec\left(F_i^T P_{p-i+1}\right)$$

$$+ \underbrace{\sum_{i=1}^{p-1}\sum_{j=0}^{p-1}}_{1\le i+j\le p} vec\left(V_{ij}R^{-1}N^T H_{p+1-i-j}\right) + \underbrace{\sum_{i=1}^{p-1}\sum_{j=0}^{p-1}}_{1\le i+j\le p} vec\left(H_{p+1-i-j}^T NR^{-1}V_{ij}^T\right)$$

$$+ \underbrace{\sum_{i,d=1}^{p-1}\sum_{j,c=0}^{p-1}}_{i+j+c+d=p+1} vec\left(V_{ij}R^{-1}V_{dc}^T\right) \tag{5.34}$$

Using Theorem 3.9, given in chapter 3, we can write

$$vec\left(P_p^T F_1\right) = U_{n\times n^p} vec\left(F_1^T P_p\right) \tag{5.35}$$

$$vec\left(P_p^T G_0 R^{-1} N^T H_1\right) = U_{n\times n^p} vec(H_1^T NR^{-1} G_0^T P_p) \tag{5.36}$$

$$vec(P_p^T G_0 R^{-1} V_{10}^T) = U_{n\times n^p} vec\left(V_{10} R^{-1} G_0^T P_p\right) \tag{5.37}$$

By replacing (5.34), (5.35), (5.36) and (5.37) in (5.33), we write

$$\left(I_{n^{p+1}} + U_{n\times n^p}\right)\left(vec(F_1^T P_p) - vec(H_1^T NR^{-1} G_0^T P_p) - vec\left(V_{10} R^{-1} G_0^T P_p\right)\right) = \mathscr{H}_p \tag{5.38}$$

Using Theorem 3.10, given in chapter 3, we have

$$vec\left(F_1^T P_p\right) = \left(I_{n^p} \otimes F_1^T\right) vec\left(P_p\right) \tag{5.39}$$

$$vec\left(H_1^T NR^{-1} G_0^T P_p\right) = \left(I_{n^p} \otimes H_1^T NR^{-1} G_0^T\right) vec\left(P_p\right) \tag{5.40}$$

$$vec\left(V_{10} R^{-1} G_0^T P_p\right) = \left(I_{n^p} \otimes V_{10} R^{-1} G_0^T\right) vec\left(P_p\right) \tag{5.41}$$

Replacing (5.39), (5.40) and (5.41) in (5.38) will lead to

$$\left(I_{n^{p+1}}+U_{n\times n^p}\right)\left(\left(I_{n^p}\otimes F_1^T\right)-\left(I_{n^p}\otimes H_1^T NR^{-1}G_0^T\right)-\left(I_{n^p}\otimes V_{10}R^{-1}G_0^T\right)\right)vec\left(P_p\right)=\mathscr{H}_p \qquad (5.42)$$

Using the distributivity property of the KP, we write

$$\left(I_{n^{p+1}}+U_{n\times n^p}\right)\left(I_{n^p}\otimes\left(F_1^T-H_1^T NR^{-1}G_0^T-P_1^T G_0 R^{-1}G_0^T\right)\right)vec\left(P_p\right)=\mathscr{H}_p \qquad (5.43)$$

If we note

$$\mathscr{F}_p=\left(I_{n^{p+1}}+U_{n\times n^p}\right)\left(I_{n^p}\otimes\left(F_1^T-H_1^T NR^{-1}G_0^T-P_1^T G_0 R^{-1}G_0^T\right)\right) \qquad (5.44)$$

Then, the equation (5.43) will be

$$\mathscr{F}_p\, vec(P_p)=\mathscr{H}_p \qquad (5.45)$$

Note from Theorem 4.3, in chapter 4, $F_1^T-H_1^T NR^{-1}G_0^T-P_1^T G_0 R^{-1}G_0^T$ is regular. The matrix $\left(I_{n^{p+1}}+U_{n\times n^p}\right)$ is regular for p even and singular for p odd (refer to chapter 3). So, to calculate P_p, two cases of calculus arise.

5.4.3.1 p even

Noting from Theorem 3.18 in chapter 3, that the matrix $\left(I_{n^{p+1}}+U_{n\times n^p}\right)$ is regular for p even, then $\mathscr{F}_p$ is regular. Hence, $\mathscr{F}_p^{-1}$ exist. From the equation (5.45), we can write

$$vec(P_p)=\mathscr{F}_p^{-1}\mathscr{H}_p \qquad (5.46)$$

Since all terms of $\mathscr{H}_p$ and $\mathscr{F}_p^{-1}$ are known, it's easy to calculate $vec(P_p)$ and then deduce P_p.

5.4.3.2 p odd

To overcome the problem of singularity, Rotella and Tanguy [3] introduce the so-called non-redundant j-power $\tilde{x}^{|j|}$ of a vector x as defined in Definition 3.6 in chapter 3. Noting that $x^{|j|}=T_j\tilde{x}^{|j|}$ and $\tilde{x}^{|j|}=T_j^{+}x^{|j|}$ where T_j is a transformation matrix and T_j^+ its pseudo inverse given in chapter 3, the equation (5.24) can be written in terms of $\tilde{x}^{|j|}$. Then, the coefficients of $\tilde{x}^{|p+1|}$ are given in (5.38) but multiplied by T_{p+1}^T on the left side, *i.e.*,

$$T_{p+1}^T\left(I_{n^{p+1}}+U_{n\times n^p}\right)\left(vec(F_1^TP_p)-vec(H_1^TNR^{-1}G_0^TP_p)-vec\left(V_{10}R^{-1}G_0^TP_p\right)\right)=T_{p+1}^T\mathscr{H}_p \quad (5.47)$$

By the linearity of the $vec(.)$ operator and using $V_{10}=P_1^TG_0$, we have

$$T_{p+1}^T\left(I_{n^{p+1}}+U_{n\times n^p}\right)vec\left(\left(F_1^T-H_1^TNR^{-1}G_0^T-P_1^TG_0R^{-1}G_0^T\right)P_p\right)=T_{p+1}^T\mathscr{H}_p \quad (5.48)$$

Also, we write

$$P_p=\tilde{P}_pT_p^+ \quad (5.49)$$

Injecting the equation (5.49) in (5.48) leads to

$$T_{p+1}^T\left(I_{n^{p+1}}+U_{n\times n^p}\right)vec\left(\left(F_1^T-H_1^TNR^{-1}G_0^T-P_1^TG_0R^{-1}G_0^T\right)\tilde{P}_pT_p^+\right)$$

$$=T_{p+1}^T\mathscr{H}_p \quad (5.50)$$

By using Theorem 3.10 given in chapter 3, (5.54) will be

$$T_{p+1}^T\left(I_{n^{p+1}}+U_{n\times n^p}\right)\left(T_p^{+T}\otimes I_n\right)vec\left(\left(F_1^T-H_1^TNR^{-1}G_0^T-P_1^TG_0R^{-1}G_0^T\right)\tilde{P}_p\right)$$
$$=T_{p+1}^T\mathscr{H}_p \quad (5.51)$$

By applying again Theorem 3.10, given in chapter 3, (5.51) will be

$$T_{p+1}^T\left(I_{n^{p+1}}+U_{n\times n^p}\right)\left(T_p^{+T}\otimes I_n\right)\left(I_{\tilde{p}}\otimes\left(F_1^T-H_1^TNR^{-1}G_0^T-P_1^TG_0R^{-1}G_0^T\right)\right)$$
$$vec\left(\tilde{P}_p\right)=T_{p+1}^T\mathscr{H}_p \quad (5.52)$$

If we note

$$\mathscr{T}_p=T_{p+1}^T\left(I_{n^{p+1}}+U_{n\times n^p}\right)\left(T_p^{+T}\otimes I_n\right) \quad (5.53)$$

$$\tilde{\mathscr{F}}_p = \mathscr{F}_p\left(I_{\sim_p} \otimes \left(F_1^T - H_1^T N R^{-1} G_0^T - P_1^T G_0 R^{-1} G_0^T\right)\right) \tag{5.54}$$

$$\tilde{\mathscr{H}}_p = T_{p+1}^T \mathscr{H}_p \tag{5.55}$$

Then, (5.52) will be written

$$\tilde{\mathscr{F}}_p vec\left(\tilde{P}_p\right) = \tilde{\mathscr{H}}_p \tag{5.56}$$

The matrix $\mathscr{F}_p$ is a rectangular matrix, of $\sim_{p+1}$ rows and $n.\sim_p$ columns, which has the property of being of full rank. Note that $\sim_p$ is the binomial coefficient as defined in chapter 3. If we note $\mathscr{F}_p^{\,+}$ the Moore-Penrose Pseudo-Inverse of $\mathscr{F}_p$, *i.e.*,

$$\mathscr{F}_p^{\,+} = \left(\mathscr{F}_p^{\,T} \mathscr{F}_p\right) \mathscr{F}_p^{\,T} \tag{5.57}$$

Hence, (5.56) becomes

$$\left(I_{\sim_p} \otimes \left(F_1^T - H_1^T N R^{-1} G_0^T - P_1^T G_0 R^{-1} G_0^T\right)\right) vec\left(\tilde{P}_p\right) = \mathscr{F}_p^{\,+} \tilde{\mathscr{H}}_p \tag{5.58}$$

Since the matrix $\left(I_{\sim_p} \otimes \left(F_1^T - H_1^T N R^{-1} G_0^T - P_1^T G_0 R^{-1} G_0^T\right)\right)$ is regular, it can be inverted and hence $vec\left(\tilde{P}_p\right)$ can be calculated. Once we have $\tilde{P}_p$, we can easily calculate P_p through (5.49).

5.5 Calculus of the feedback control

By substituting (5.7), (5.9) and (5.10) into (5.5), we write

$$u^* = -R^{-1}\left[N^T \sum_{i=1}^{h} H_i x^{[i]} + \left(\sum_{j=0}^{g}\left(I_n \otimes x^{[j]T}\right) G_j^T\right)\left(\sum_{k=1}^{\bar{p}} P_k x^{[k]}\right)\right] \tag{5.59}$$

that is,

$$u^* = -\sum_{i=1}^{h} R^{-1} N^T H_i x^{[i]} - \sum_{i=1}^{\bar{p}} \sum_{j=0}^{g} R^{-1} \left(I_n \otimes x^{[j]T} \right) G_j^T P_i x^{[i]} \tag{5.60}$$

Applying Lemmas 3.2 and 3.3, introduced in chapter 3, we obtain

$$u^* = -\sum_{i=1}^{h} R^{-1} N^T H_i x^{[i]} - \sum_{i=1}^{\bar{p}} \sum_{j=0}^{g} R^{-1} mat^T_{n^{i+j} \times m} \left(vec\left(P_i^T G_j \right) \right) x^{[i+j]} \tag{5.61}$$

Hence, we can write

$$u^* = \sum_{k \geq 1} u_k x^{[k]} \tag{5.62}$$

where

$$u_k = -R^{-1} \left[N^T H_k + \underbrace{\sum_{i=1}^{k} \sum_{j=0}^{k-1}}_{i+j=k} mat^T_{n^t \times m} \left(vec\left(P_i^T G_j \right) \right) \right] \tag{5.63}$$

or equivalently,

$$u_k = -R^{-1} \left[N^T H_k + \underbrace{\sum_{i=1}^{k} \sum_{j=0}^{k-1}}_{i+j=k} V_{ij}^T \right] \tag{5.64}$$

For the design of the suboptimal control based on the KP introduced above, as given by Rotella and Tanguy [3], the stability of the closed loop system is not ensured and not treated. Khayati and Benabdelkader [57] extend this work to a new approach based on the Lyapunov Function to design a suboptimal control which ensures the stability within an interval of attraction. This new approach will be presented and discussed in chapter 6.

5.6 Application to nonlinear scalar models

Example 5.1: Consider the tutorial example discussed in [3] defined by

$$\dot{x} = -1.4x - 0.5x^2 + (0.5 - 2.5x)\cdot u \tag{5.65}$$

where $x \in \mathbb{R}$ and $u \in \mathbb{R}$ are the state and the control input, respectively. The cost functional to be optimized is defined by

$$J = \frac{1}{2}\int_0^\infty \left(2.4x^2 + 2.5u^2\right)dt \tag{5.66}$$

Referring to [3] the "exact" optimal control is

$$u^* = \frac{7x + 2.5x^2 - x\sqrt{156.25x^2 - 25x + 55}}{12.5(0.2 - x)} \tag{5.67}$$

This complex control function is hardly practical and unbounded for $x = 0.2$. The design of the sub-optimal control till the 2nd order leads to the computation of the P_i and the u_i as follows

$$P(x) = 0.8324x - 0.047x^2 \tag{5.68}$$

$$\bar{u} = -0.1664x + 0.8418x^2 \tag{5.69}$$

The design and simulation of the exact and sub-optimal control law of the 2nd order leads to the exact cost J^* and the suboptimal cost $\bar{J}$. The results are presented in Table 5.1 for different initial conditions, $x(0)$, in terms of relative cost errors $£_{J(n)} = \left|\frac{\bar{J} - J^*}{J^*}\right|$ in %, and different truncation orders $n = 1$ (*i.e.*, linear control), and $n = 2$.

The results show that the cost errors relative to the exact design are much higher with the linear design than the second and third order designs. In fact, the linear approximation of the state feedback does not take into account the nonlinearities. Also, Figures 5.1 and 5.2 show the evolution of the state variable x and the control variable u *vs.* time for an initial condition $x(0) = 6$.

Table 5.1 Exact cost and Sub-optimal costs errors *vs.* Initial condition for the scalar example (5.65)

$x(0)$	J^*	$£_{J(1)}$	$£_{J(2)}$
1.0	0.3163	26.84	1.45
2.0	0.9112	70.01	13.28
3.0	1.5857	112.52	32.55
4.0	2.2954	152.68	55.07
5.0	3.0250	190.40	79.14
6.0	3.7673	225.85	104.06
7.0	4.5184	259.23	129.48
8.0	5.2759	290.74	155.21
9.0	6.0385	320.53	181.14
10.0	6.8049	348.77	207.22

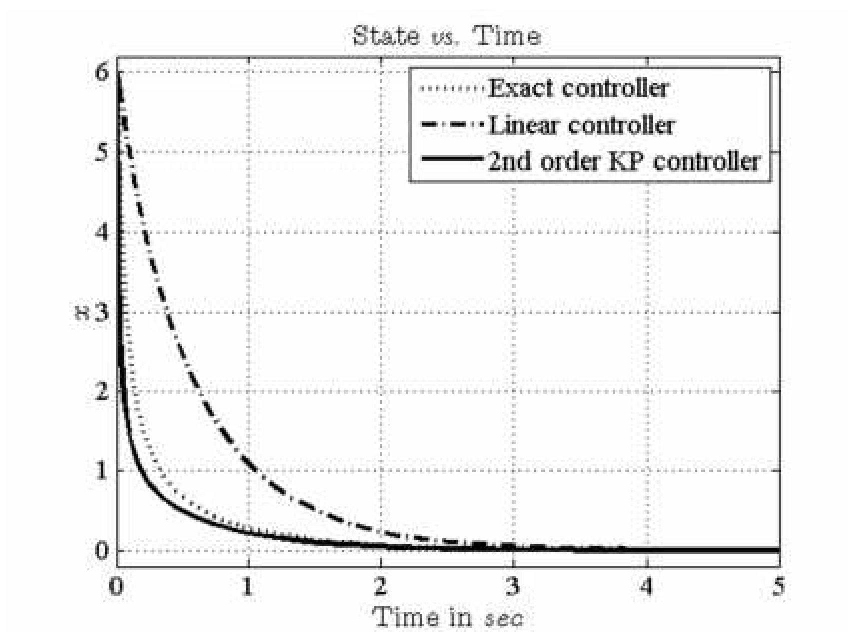

Figure 5.1 State evolution for scalar example (5.65)

The simulations of the state evolution for both controllers show that the exact and the sub-optimal controllers present almost the same behaviour.

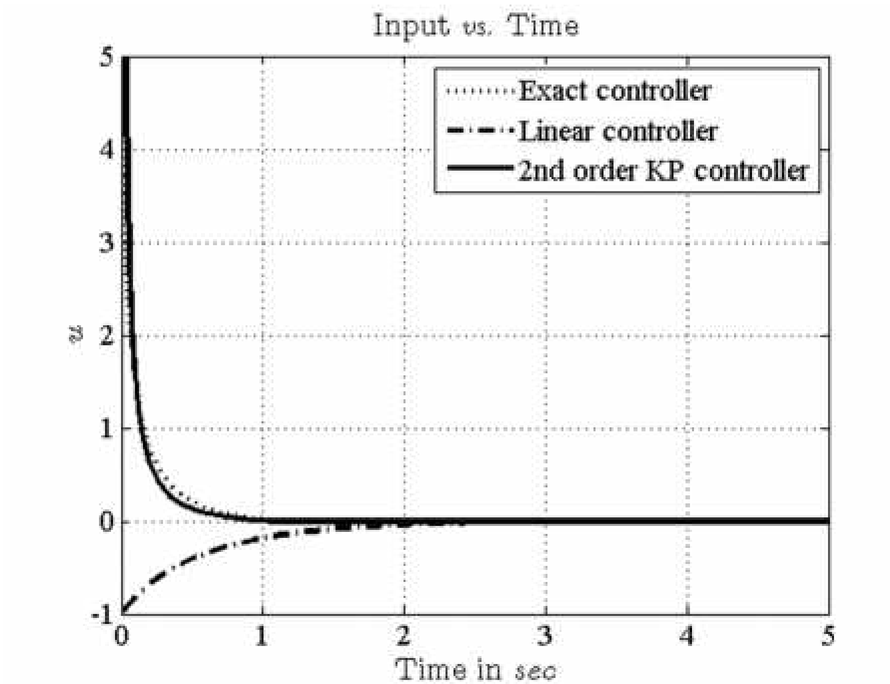

Figure 5.2 Input signal evolution for scalar example (5.65)

The simulations of the control inputs show again that the exact and the sub-optimal controllers present the same behaviour.

Moreover, note that the simulation of the sub-optimal controls for the initial condition $x(0) = 0.2$, show the following costs $\overline{J}_{(n=1)} = 0.01652$, $\overline{J}_{(n=2)} = 0.01629$ whereas the exact optimal control fails to stabilize the system for this particular initial condition.

Example 5.2: Consider the tutorial example discussed in [58] defined by

$$\dot{x} = x - x^3 + u \tag{5.70}$$

where $x \in \mathbb{R}$ and $u \in \mathbb{R}$ are the state and the control input, respectively. The cost function to be optimized is defined by

$$J=\frac{1}{2}\int_0^{\infty}\left(x^2+u^2\right)dt \qquad (5.71)$$

From (5.10), we obtain

$$u^*=-P\left(x\right) \qquad (5.72)$$

where $P\left(x\right)$ is solution of the state dependent equation

$$P\left(x\right)^2-2\left(x-x^3\right)P\left(x\right)-x^2=0 \qquad (5.73)$$

An exact solution of (5.73) is computed using MAPLE software as

$$P\left(x\right)=x\left(1-x^2+\sqrt{x^4-2x^2+2}\right)=x\left(1-x^2+\sqrt{\left(x^2-1\right)^2+1}\right) \qquad (5.74)$$

Hence, using (5.72), the "exact" optimal control is

$$u^*=x\left(x^2-1-\sqrt{x^3-2x^2+2}\right) \qquad (5.75)$$

The design of the sub-optimal control till the 3rd order leads to the following expression

$$\overline{u}=-\left(1+\sqrt{2}\right)x+\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)x^3 \qquad (5.76)$$

The design and simulation of the discussed (exact and sub-optimal) techniques lead to the exact cost J^* and the suboptimal cost $\overline{J}$ of 1st and 3rd order of truncation. The results are presented in Table 5.2 for different initial conditions, $x\left(0\right)$, in terms of relative cost errors, $£_{J(n)}=\left|\frac{\overline{J}-J^*}{J^*}\right|$ in %.

The results of the different simulations show that below an initial condition of 1.2, the cost errors *w.r.t.* the exact design are higher with the linear (1st order) controller than those of 3rd order controller. The 3rd order design represents a good estimation of the exact controller since the errors are less than 5%. Figures 5.3 and 5.4 shown below are respectively the evolution in time of the state variable x and the input control u for the three different controllers (exact, linear and 3^{rd} order) with an initial condition of 1.

Table 5.2 Exact cost and sub-optimal cost error *vs.* Initial condition for the scalar example (5.70)

$x(0)$	J^*	$£_{Lin\,\&\,2^{nd}}$	$£_{3^{rd}\,\&\,4^{th}}$
0.4	0.18	0.38	0
0.6	0.38	1.65	0
0.8	0.61	4.87	0.05
1.0	0.82	10.81	0.56
1.2	1.00	19.66	5.89
1.4	1.14	30.29	115.98

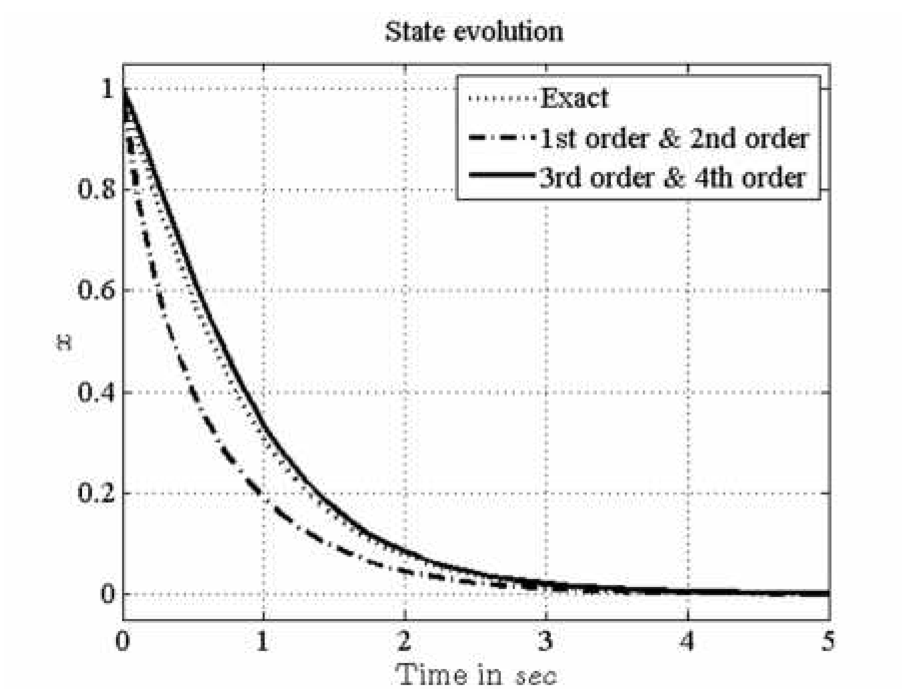

Figure 5.3 State evolution for the scalar example (5.70)

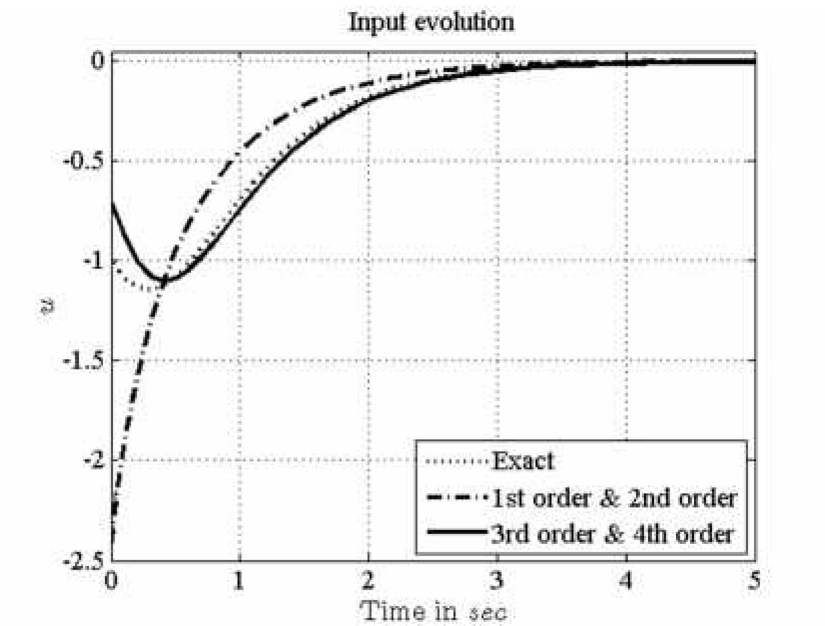

Figure 5.4 Input signal evolution for the scalar example (5.70)

We note from Figures 5.3 and 5.4 that the 3rd order design shows a better curve fitting *w.r.t.* the exact design than the linear one in terms of both the state variable and input control. This improvement comes from the fact that we have a better function estimation by the introduction of the nonlinearities in the estimation process.

Example 5.3: Consider the scalar example defined by

$$\dot{x} = x + x^2 + u \tag{5.77}$$

where $x \in \mathbb{R}$ and $u \in \mathbb{R}$ are the state and the control input, respectively. The cost function to be optimized is defined by

$$J = \frac{1}{2}\int_0^\infty \left(x^2 + u^2\right) dt \tag{5.78}$$

Note that the autonomous part of the dynamics (*i.e.*, unforced system) is unstable. from (5.10), the optimal control is

$$u^* = -P\left(x\right) \tag{5.79}$$

where $P(x)$ is solution of the equation

$$P(x)^2 - 2(x+x^2)P(x) - x^2 = 0 \tag{5.80}$$

The resolution of the equation (5.90) leads to

$$P(x) = x + x^2 + \sqrt{(x+x^2)^2 + x^2} \tag{5.81}$$

Hence, the "exact" optimal control is

$$u^* = -\left(x + x^2 + \sqrt{(x+x^2)^2 + x^2} \right) \tag{5.82}$$

The design of the sub-optimal control of the 3rd order leads to

$$\bar{u} = 2x + x^2 + \left(\frac{21}{8} - \frac{3}{\sqrt{2}} \right) x^3 \tag{5.83}$$

The design and simulation of the proposed (exact and sub-optimal) techniques lead to the exact cost J^* and the suboptimal cost $\bar{J}$. The results are presented in Table 5.3 for different initial conditions, $x(0)$, in terms of the relative cost errors, $£_{J(n)} = \left| \frac{\bar{J} - J^*}{J^*} \right|$, in % with the selected truncation orders one, two and three.

The simulation results show that below an initial condition of 1.0, the linear controller works but with errors much higher than the nonlinear ones (of 2nd and 3rd orders). For an initial condition higher than 1.0, the linear controller cannot stabilize the system, in contrast the nonlinear ones are stabilizing ones, with the advantage for the 3rd order controller having smaller relative errors. Figures 5.5 and 5.6 are respectively the evolution of the state variable x and the input control u for the different controllers (exact, linear, 2nd order and 3rd order ones) for an initial condition of 0.8.

Table 5.3 Exact Cost and Sub-optimal costs errors *vs.* Initial condition for the scalar example (5.77)

$x(0)$	J^*	$£_{Lin}$	$£_{2^{nd}}$	$£_{3^{rd}}$
0.1	0.01	28.79	25.55	25.39
0.2	0.05	34.95	27.01	26.54
0.3	0.12	43.28	28.88	27.75
0.4	0.23	54.40	30.97	28.89
0.5	0.37	69.64	33.39	29.95
0.6	0.56	91.28	36.13	30.97
0.7	0.79	123.32	39.15	31.85
0.8	1.07	176.50	42.43	32.77
0.9	1.41	286.12	46.07	33.54
1.0	1.80	Unst.	49.91	34.14
1.1	2.26	Unst.	54.11	34.74
1.2	2.78	Unst.	58.62	35.29
1.3	3.37	Unst.	63.50	35.87
1.4	4.03	Unst.	68.70	36.33
1.5	4.77	Unst.	74.21	36.80
1.6	5.59	Unst.	80.11	37.27
1.7	6.49	Unst.	86.45	37.73
1.8	7.48	Unst.	93.10	38.22
1.9	8.57	Unst.	100.14	38.67
2.0	9.75	Unst.	107.64	39.18

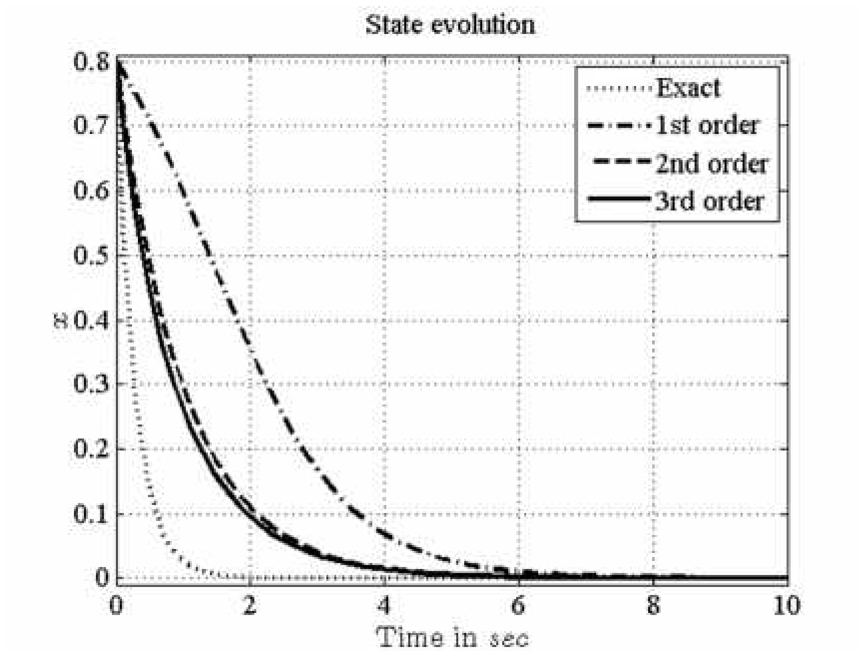

Figure 5.5 State evolution for the scalar example (5.77)

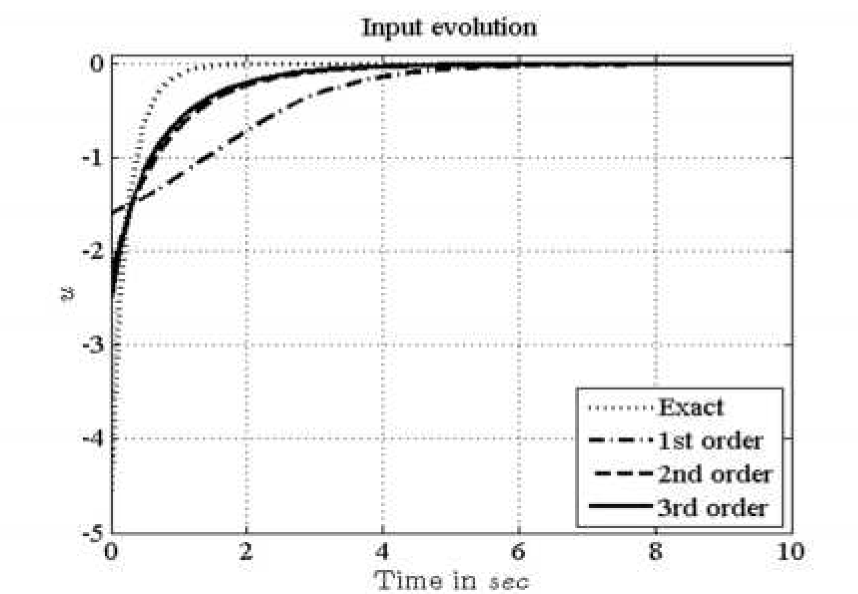

Figure 5.6 Input signal evolution for the scalar example (5.77)

The simulations of the exact, linear and (2^{nd} and 3^{rd} order) nonlinear controllers, with an initial condition of 0.8, show that for both the state and input control responses, the best approximation of the exact controller behaviour is given by the nonlinear ones (2^{nd} and 3^{rd} order controllers) with a slightly better advantage for the 3^{rd} order one.

5.7 Conclusion

In this chapter, we presented the optimal control of polynomial systems using the KP method. In section 5.2, we stated the problem which leads to the equation to be solved. In section 5.3, we made an approximation of the unknown terms into a polynomial form in terms of the KP, which leads to the resolution of uncoupled linear equations. The resolution of these equations is presented in section 5.4 leading to the calculus of the feedback optimal control law. In section 5.5, we showed the application of this method to three scalar examples. Despite the fact that this method enlarges the interval of attraction with higher order of truncation in the equation of approximation, it does not guarantee automatically the stability of the system. This stability will be guaranteed by the new method so-called KP-Lyapunov-function-based control that will be presented in the next chapter.

6 Optimal control using Kronecker product Lyapunov function based technique

6.1 Introduction

In chapter 5, we presented the optimal control method using the KP-based polynomial expansion, which has the advantage of a larger domain of attraction compared with the linear techniques. But, despite this advantage, this method has a limitation. In fact, it does not guarantee the stability of the closed loop system since the computation of the cost function $V(x)$ does not satisfy the conditions of the stability. Alternatively, we will propose a new method by choosing $V(x)$ in a quadratic form to satisfy the conditions of a Lyapunov candidate function and then guaranteeing the global asymptotical stability (GAS) in the sense of Lyapunov, eventually [57].

This new method will be the aim of chapter 6. After introducing this chapter, we will state, in section 6.2, the optimal control problem. In section 6.3, we will transform the problem into a system of uncoupled linear equations and we will choose the cost function $V(x)$ in appropriate form to ensure the GAS. The resolution of these linear equations for a given order of truncation will be showed in section 6.4. For each order, we will present the algorithm to calculate the different gain matrices. Based on the calculation of these gains, we will present in section 6.5 the state feedback design which leads to the sub-optimal control law using the KP-Lyapunov-Function (LF) technique. In section 6.6, we will check the stability of the closed loop system. In section 6.7, we will illustrate the improvement in terms of control performance through two nonlinear plants: a scalar example and the F8 fighter model. For both systems, we will run the simulations and compare the results using three techniques Linear, KP, KP-LF based ones. Finally in the conclusion, we will summarize the sections of this chapter and the main contributions in terms of stability framework.

6.2 Statement of the problem

Let's consider the nonlinear dynamics

$$\dot{x}(t) = F(x) + G(x)\cdot u(t) = F(x) + \sum_{k=1}^{m} G_k(x)\cdot u_k(t) \tag{6.1}$$

where $t \in \mathbb{R}$ is the time, $x(t) \in \mathbb{R}^n$ the state vector, $u(t) = \left[u_1(t) \quad .... \quad u_m(t)\right]^T \in \mathbb{R}^m$ is the input vector. $F(.), G_k(.)$, for $k = 1, \ldots, m$, are analytic vector fields from $\mathbb{R}^n$ into $\mathbb{R}^n$. Note that $G(x) = \left[G_1(x) \ |....| \ G_m(x)\right] \in \mathbb{R}^{n\times m}$. We write

$$F(x) = \sum_{j\geq 1} F_j \cdot x^{|j|} \tag{6.2}$$

$$G_k(x) = \sum_{j\geq 0} G_{kj} \cdot x^{|j|} \qquad \forall\, k = 1,\ldots,m \tag{6.3}$$

$$G(x) = \sum_{j\geq 0} G_j \left(I_m \otimes x^{|j|}\right) \tag{6.4}$$

with $F_j \in \mathbb{R}^{n\times n^j}$, $G_{kj} \in \mathbb{R}^{n\times n^j}\, \forall k = 1,\ldots,m$ and $G_j = \left[G_{1j} \quad |....| \quad G_{mj}\right] \in \mathbb{R}^{n\times mn^j}$. Let $z(t) = H(x) \in \mathbb{R}^q$ be a vector function of the states given by

$$H(x) = \sum_{j\geq 1} H_j \cdot x^{|j|} \tag{6.5}$$

The problem of optimal control is to design a state feedback which minimizes the continuous time cost functional

$$J = \frac{1}{2}\int_0^{\infty} \left[z(t)^T Q z(t) + u(t)^T R u(t) \right] dt \tag{6.6}$$

where Q is a non-negative definite matrix of $\mathbb{R}^{q\times q}$ and R is a positive definite matrix of $\mathbb{R}^{m\times m}$. We denote by $V(x)$ the optimal cost with an initial condition x at t [42].

$$V(x) = \frac{1}{2}\int_t^{\infty} \left[z(\tau)^T Q z(\tau) + u^*(\tau)^T R u^*(\tau) \right] d\tau \tag{6.7}$$

where $u^* = \arg\left(\min_u J\right)$ is the optimal control. As widely discussed in chapter 4, the optimality conditions are given as follows [42]

$$u^*(x) = -R^{-1}G(x)^T V_x(x) \tag{6.8}$$

$$H(x)^T QH(x) + V_x(x)^T F(x) + F(x)^T V_x(x) - V_x(x)^T G(x) R^{-1} G(x)^T V_x(x) = 0 \tag{6.9}$$

where $V_x(x) = \dfrac{\partial V}{\partial x}$ denotes the derivative of $V(x)$ *w.r.t.* the state vector x.

6.3 Equation of approximation

Based on the optimality condition (6.8) and (6.9), the design of the optimal state feedback that will be discussed later is proposed in polynomial form using the KP tensor, the *vec* and *mat* notations [3-57]. This design is based on the determination of the cost function $V(x)$ presented in a quadratic form. According to [5], $V(x)$ would be expected to satisfy the conditions of any Lyapunov candidate function. Let's consider the $V(x)$ in form

$$V(x) = \frac{1}{2}\begin{pmatrix} x^T & \sum_{j\geq 2} x^{|j|T} \cdot P_j^T \end{pmatrix} \begin{pmatrix} P & \sim I_n \\ \sim I_n & I_n \end{pmatrix} \begin{pmatrix} x \\ \sum_{j\geq 2} P_j \cdot x^{|j|} \end{pmatrix} \tag{6.10}$$

with $\sim \in \mathbb{R}$, $P = P^T > 0$ in $\mathbb{R}^{n\times n}$ and P_j; $j \geq 2$; constant matrices of $\mathbb{R}^{n\times n^j}$. Assuming that P is a symmetric positive definite matrix, and using the Cholesky decomposition, it exists P_1 in $\mathbb{R}^{n\times n}$ such that $P = P_1^T P_1$. Then, by substituting (6.10) in (6.9) and replacing P by $P_1^T P_1$, the equation (6.10) can be written as

$$V(x) = \frac{1}{2} x^T P_1^T P_1 x + \frac{\sim}{2}\sum_{i\geq 2} x^{|i|T} P_i^T x + \frac{\sim}{2}\sum_{j\geq 2} x^T P_j x^{|j|} + \frac{1}{2}\sum_{i\geq 2}\sum_{j\geq 2} x^{|i|T} P_i^T P_j x^{|j|} \tag{6.11}$$

If we note

$$P_{i(j)} = \begin{cases} P_1 & if \quad i = j = 1 \\ \sim I_n & if \quad i = 1,\ j \geq 2 \\ P_j & if \quad i \geq 2,\ j \geq 1 \end{cases} \tag{6.12}$$

Then $V(x)$ can be written as

$$V(x) = \frac{1}{2}\sum_{i=1}^{1}\sum_{j=1}^{1} x^{|i|T} P_{i(j)}^T P_{j(i)} x^{|j|} + \frac{1}{2}\sum_{i\geq 2}\sum_{j=1}^{1} x^{|i|T} P_{i(j)}^T P_{j(i)} x^{|j|} + \frac{1}{2}\sum_{i=1}^{1}\sum_{j\geq 2} x^{|i|T} P_{i(j)}^T P_{j(i)} x^{|j|}$$
$$+\frac{1}{2}\sum_{i\geq 2}\sum_{j\geq 2} x^{|i|T} P_{i(j)}^T P_{j(i)} x^{|j|} \tag{6.13}$$

Hence, $V(x)$ is written in a compact form as

$$V(x) = \frac{1}{2}\sum_{i\geq 1}\sum_{j\geq 1} x^{|i|T} P_{i(j)}^T P_{j(i)} x^{|j|} \tag{6.14}$$

The equation of $V(x)$ given by (6.14), where $P_{i(j)}$ is introduced in (6.12) will be advantageous to solve the nonlinear equation (6.9) in V_x. By applying the derivative of the equation (6.13) $w.r.t.$ x, we have

$$V_x = \frac{\partial V}{\partial x} = \sum_{i=1}\sum_{j=1} \frac{\partial x^{|j|T}}{\partial x} P_{j(i)}^T P_{i(j)} x^{|i|} \tag{6.15}$$

Referring to Lemma 3.1 introduced in chapter 3 to write

$$\frac{\partial x^{|j|}}{\partial x^T} = D_j^{(n)} \cdot \left(I_n \otimes x^{|j-1|}\right) \tag{6.16}$$

where $D_j^{(n)}$ is the square j-differential Kronecker matrix of $\mathbb{R}^{n^j \times n^j}$ introduced in chapter 3. Thus, the expression (6.15) will be

$$V_x = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left(I_n \otimes x^{|j-1|T}\right) D_j^{(n)T}\ P_{j(i)}^T P_{i(j)} x^{|i|} \tag{6.17}$$

Applying Lemmas 3.2 and 3.3 introduced in chapter 3, to the expression $\left(I_n \otimes x^{|j-1|T}\right) D_j^{(n)T}\ P_{j(i)}^T P_{i(j)} x^{|i|}$ leads to

$$\left(I_n \otimes x^{|j-1|T}\right) D_j^{(n)T}\ P_{j(i)}^T P_{i(j)} x^{|i|} = \left(I_n \otimes vec^T\left(P_{i(j)}{}^T P_{j(i)} D_j^{(n)}\right)\right)\left(vec\left(I_n \otimes I_{n^i n^{j-1}}\right)\right) x^{|i+j-1|}$$

$$= mat^T_{n^{i+j-1}\times n}\left(vec\left(P_{i(j)}{}^T P_{j(i)} D_j^{(n)}\right)\right) x^{|i+j-1|} \qquad (6.18)$$

We note

$$V_{ij} = mat^T_{n^{i+j-1}\times n}\left(vec\left(P_{i(j)}{}^T P_{j(i)} D_j^{(n)}\right)\right) \quad \in \mathbb{R}^{n\times n^{i+j-1}} \qquad (6.19)$$

Hence, the equation (6.17) will be

$$V_x = \sum_{i\geq 1}\sum_{j\geq 1} V_{ij} x^{|i+j-1|} \qquad (6.20)$$

Injecting the equations (6.2), (6.4), (6.5) and (6.20) into (6.9) leads to

$$\sum_{i,j,k\geq 1} x^{|i+j-1|T} V_{ij}^T F_k x^{|k|} + \sum_{i,j,k\geq 1} x^{|k|T} F_k^T V_{ij} x^{|i+j-1|} + \sum_{i,j\geq 1} x^{|i|T} H_i^T Q H_j x^{|k|}$$

$$-\left[\sum_{i,j\geq 1}\sum_{k\geq 0} x^{|i+j-1|T} V_{ij}^T G_k \left(I_n \otimes x^{|k|}\right)\right] R^{-1} \left[\sum_{b,c\geq 1}\sum_{d\geq 0}\left(I_n \otimes x^{|b|T}\right) G_d^T V_{bc} x^{|b+c-1|}\right] = 0 \qquad (6.21)$$

By using Theorem 3.14 and Lemma 3.3, we have

$$\sum_{i,j,k\geq 1} vec^T\left(V_{ij}^T F_k\right) x^{|i+j+k-1|} + \sum_{i,j,k\geq 1} vec^T\left(F_k^T V_{ij}\right) x^{|i+j+k-1|} + \sum_{i,j\geq 1} vec^T\left(H_i^T Q H_j\right) x^{|i+j|}$$

$$-\left[\sum_{i,j\geq 1}\sum_{k\geq 0} mat^T_{n^{i+j+k-1}\times m}\left(vec\left(V_{ij}^T G_k\right) x^{|i+j+k-1|}\right)\right] R^{-1} \left[\sum_{b,c\geq 1}\sum_{d\geq 0} x^{|b+c+d-1|}\ mat^T_{n^{b+c+d-1}\times m}\left(vec\left(V_{bc}^T G_d\right)\right)\right] = 0 \qquad (6.22)$$

If we note

$$W_{ijk} = mat^T_{n^{i+j+k-1}\times m}\left(vec\left(V_{ij}^T G_k\right)\right) \quad \in \mathbb{R}^{m\times n^{i+j+k-1}} \qquad (6.23)$$

Hence, the equation (6.22) will be

$$\sum_{i,j,k\geq 1} vec^T\left(V_{ij}^T F_k\right) x^{|i+j+k-1|} + \sum_{i,j,k\geq 1} vec^T\left(F_k^T V_{ij}\right) x^{|i+j+k-1|} + \sum_{i,j\geq 1} vec^T\left(H_i^T Q H_j\right) x^{|i+j|}$$

$$-\sum_{i,j,b,c\geq 1}\sum_{k,d\geq 0} vec^T\left(W_{ijk} R^{-1} W_{bcd}\right) x^{|i+j+k+b+c+d-2|} = 0 \qquad (6.24)$$

In order to obtain the optimal control u^*, we have to find the polynomial $V(x)$. In other words, we have to calculate the different terms P_i, $i \in \mathbb{N}^*$. The terms P_i are obtained by cancelling the coefficients of $x^{|i+1|}$ in (6.24). This procedure is the subject of the following section.

6.4 Determination of P_i

6.4.1 First order

The calculation of P_1 from (6.24) is given by the cancellation of the coefficients of $x^{|2|}$. Noting that the first differential Kronecker matrix is given by $D_1^{(n)} = I_n$ and that $P = P_1^T P_1$, we use (6.13), (6.19), (6.23), (6.24) and the *mat* notation to obtain

$$vec\left(PF_1\right) + vec\left(F_1^T P\right) + vec\left(H_1^T Q H_1\right) - vec\left(P G_0 R^{-1} G_0^T P\right) = 0 \qquad (6.25)$$

Since the operator $vec(.)$ is linear *w.r.t.* the matrices of the same dimensions, the equation (6.25) will be

$$PF_1 + F_1^T P + H_1^T Q H_1 - P G_0 R^{-1} G_0^T P = 0 \qquad (6.26)$$

The equation (6.26) is the classical ARE and P_1 given from $P = P_1^T P_1$ is the gain matrix of the optimal linear controller for the linearized system.

6.4.2 Second order

The calculation of P_2 from (6.24) is given by the cancellation of the coefficients of $x^{|3|}$. We obtain

$$vec\left(V_{21}^T F_1\right) + vec\left(V_{12}^T F_1\right) + vec\left(V_{11}^T F_2\right) + vec\left(F_1^T V_{21}\right) + vec\left(F_1^T V_{12}\right) + vec\left(F_2^T V_{11}\right)$$
$$+vec\left(H_1^T Q H_2\right) + vec\left(H_2^T Q H_1\right) - vec\left(W_{210}^T R^{-1} W_{110}\right) - vec\left(W_{120}^T R^{-1} W_{110}\right)$$
$$-vec\left(W_{111}^T R^{-1} W_{110}\right) - vec\left(W_{110}^T R^{-1} W_{210}\right) - vec\left(W_{110}^T R^{-1} W_{120}\right) - vec\left(W_{110}^T R^{-1} W_{111}\right) = 0 \qquad (6.27)$$

By using (6.13), (6.19) and (6.22), noting that $D_1^{(n)} = I_n$, and applying Theorem 3.9 , Theorem 3.10 and the *mat* notation given in chapter 3, the equation (6.27) will be

$$\left(F_1^T \otimes I_n\right) U_{n\times n^2} \sim vec\left(P_2\right) + \left(F_1^T \otimes I_{n^2}\right)\left(D_2^{(n)T} \otimes I_n\right) vec\left(P_2\right) \sim + vec\left(PF_2\right)$$
$$+U_{n^2\times n}\left(F_1^T \otimes I_{n^2}\right) U_{n\times n^2} \sim vec\left(P_2\right) + U_{n^2\times n}\left(F_1^T \otimes I_{n^2}\right)\left(D_2^{(n)T} \otimes I_n\right) vec\left(P_2\right) \sim$$
$$+vec\left(F_2^T P\right) U_{n^2\times n} + vec\left(H_2^T Q H_1\right) + vec\left(H_2^T Q H_1\right) - \left(\left(P G_0 R^{-1} G_0^T\right) \otimes I_{n^2}\right) U_{n\times n^2} \sim vec\left(P_2\right)$$
$$-U_{n^2\times n}\left(\left(P G_0 R^{-1} G_0^T\right) \otimes I_{n^2}\right) U_{n\times n^2} vec\left(P_2\right) \sim - \left(\left(P G_0 R^{-1} G_0^T\right) \otimes I_{n^2}\right)\left(D_2^{(n)T} \otimes I_n\right) \sim vec\left(P_2\right)$$
$$-U_{n^2\times n}\left(\left(P G_0 R^{-1} G_0^T\right) \otimes I_{n^2}\right)\left(D_2^{(n)T} \otimes I_n\right) \sim vec\left(P_2\right) - vec\left(P G_1\left(I_n \otimes R^{-1} G_0 P\right)\right)$$
$$-U_{n^2\times n} vec\left(P G_1\left(I_n \otimes R^{-1} G_0 P\right)\right) = 0 \qquad (6.28)$$

In fact, we have

$$vec\left(V_{21}^T F_1\right) = vec\left[mat_{n^2\times n}\left(vec\left(P_{2(1)}^T P_{1(2)} D_1^{(n)}\right)\right) F_1\right]$$
$$= vec\left[mat_{n^2\times n}\left(vec\left(P_2^T \sim I_n I_n\right)\right) F_1\right]$$
$$= \sim vec\left[mat_{n^2\times n}\left(vec\left(P_2^T\right)\right) F_1\right]$$
$$= \sim vec\left(P_2^T F_1\right)$$
$$= \sim \left(F_1^T \otimes I_{n^2}\right) vec\left(P_2^T\right)$$
$$= \sim \left(F_1^T \otimes I_{n^2}\right) U_{n\times n^2} vec\left(P_2\right) \qquad (6.29)$$

$$vec\left(V_{12}^T F_1\right) = vec\left[mat_{n^2\times n}\left(vec\left(P_{1(2)}^T P_{2(1)} D_2^{(n)}\right)\right)F_1\right]$$
$$= vec\left[mat_{n^2\times n}\left(vec\left(\sim I_n P_2 D_2^{(n)}\right)\right)F_1\right]$$
$$= \sim\ vec\left[mat_{n^2\times n}\left(vec\left(P_2 D_2^{(n)}\right)\right)F_1\right]$$
$$= \sim\left(F_1^T \otimes I_{n^2}\right)vec\left[mat_{n^2\times n}\left(vec\left(P_2 D_2^{(n)}\right)\right)\right]$$
$$= \sim\left(F_1^T \otimes I_{n^2}\right)vec\left(P_2 D_2^{(n)}\right)$$
$$= \sim\left(F_1^T \otimes I_{n^2}\right)\left(D_2^{(n)T} \otimes I_n\right)vec\left(P_2\right) \tag{6.30}$$

$$vec\left(F_2^T V_{11}\right) = vec\left[F_2^T\ mat_{n\times n}^T\left(vec\left(P_{1(1)}^T P_{1(1)} D_1^{(n)}\right)\right)\right]$$
$$= vec\left[F_2^T\ mat_{n\times n}^T\left(vec\left(P_{1(1)}^T P_{1(1)} I_n\right)\right)\right]$$
$$= vec\left[F_2^T\ mat_{n\times n}^T\left(vec\left(P\right)\right)\right]$$
$$= vec\left(F_2^T P\right) \tag{6.31}$$

$$vec\left(W_{210}^T R^{-1} W_{110}\right) = vec\left[mat_{n^2\times m}\left(vec\left(V_{21}^T G_0\right)\right)R^{-1}\ mat_{n\times m}^T\left(vec\left(V_{11}^T G_0\right)\right)\right]$$
$$= vec\left(V_{21}^T G_0 R^{-1} G_0^T V_{11}\right)$$
$$= vec\left(\sim P_2^T G_0 R^{-1} G_0^T P\right)$$
$$= \sim\left(\left(P G_0 R^{-1} G_0^T\right) \otimes I_{n^2}\right)vec\left(P_2^T\right)$$
$$= \sim\left(\left(P G_0 R^{-1} G_0^T\right) \otimes I_{n^2}\right)U_{n\times n^2}\, vec\left(P_2\right) \tag{6.32}$$

$$vec\left(W_{120}^T R^{-1} W_{110}\right) = vec\left[mat_{n^2\times m}\left(vec\left(V_{12}^T G_0\right)\right)R^{-1}\ mat_{n\times m}^T\left(vec\left(V_{11}^T G_0\right)\right)\right]$$
$$= vec\left(V_{12}^T G_0 R^{-1} G_0^T V_{11}\right)$$
$$= vec\left(V_{12}^T G_0 R^{-1} G_0^T P\right)$$
$$= \sim\left(\left(P G_0 R^{-1} G_0^T\right) \otimes I_{n^2}\right)\left(D_2^{(n)T} \otimes I_n\right)vec\left(P_2\right) \tag{6.33}$$

and

$$
\begin{aligned}
vec\left(W_{111}^{T}R^{-1}W_{110}\right) &= vec\left[mat_{n^2\times m}\left(vec\left(V_{11}^{T}G_1\right)\right)R^{-1}\ mat_{n\times m}^{T}\left(vec\left(V_{11}^{T}G_0\right)\right)\right] \\
&= vec\left(mat_{n^2\times m}\left(vec\left(V_{11}^{T}G_1\right)\right)R^{-1}\left(V_{11}^{T}G_0\right)^{T}\right) \\
&= vec\left(mat_{n^2\times m}\left(vec\left(PG_1\right)\right)R^{-1}G_0P\right) \\
&= \left(\left(PG_0^{T}R^{-1}\right)\otimes I_{n^2}\right)vec\left(mat_{n^2\times m}\left(vec\left(PG_1\right)\right)\right) \\
&= \left(\left(PG_0^{T}R^{-1}\right)\otimes I_{n^2}\right)vec\left(PG_1\right) \\
&= \left(\left(\left(PG_0^{T}R^{-1}\right)\otimes I_n\right)\otimes I_n\right)vec\left(PG_1\right) \\
&= vec\left(PG_1\left(\left(PG_0^{T}R^{-1}\right)\otimes I_n\right)^{T}\right) \\
&= vec\left(PG_1\left(I_n\otimes\left(R^{-1}G_0P\right)\right)\right) \qquad (6.34)
\end{aligned}
$$

The remaining terms of (6.27) are combined with (6.29) to (6.34) using Theorem 3.9 given in chapter 3. Then, we obtain

$$
\begin{aligned}
&\left(U_{n^2\times n}+I_{n^3}\right)\Big[vec\left(F_2^{T}P\right)+\left(F_1^{T}\otimes I_{n^2}\right)\sim U_{n\times n^2}vec\left(P_2\right) \\
&+\left(F_1^{T}\otimes I_{n^2}\right)\left(D_2^{(n)T}\otimes I_n\right)\sim vec\left(P_2\right)+vec\left(H_2^{T}QH_1\right)- \\
&\left(\left(PG_0R^{-1}G_0^{T}\right)\otimes I_{n^2}\right)\sim U_{n\times n^2}vec\left(P_2\right)-\left(\left(PG_0R^{-1}G_0^{T}\right)\otimes I_{n^2}\right)\sim\left(D_2^{(n)T}\otimes I_n\right)vec\left(P_2\right) \\
&-vec\left(\left(I_n\otimes\left(PG_0^{T}R^{-1}\right)\right)G_1^{T}P\right)\Big]=0 \qquad (6.35)
\end{aligned}
$$

Based on Theorem 3.18, introduced in chapter 3, we notice that $\left(U_{n^2\times n}+I_{n^3}\right)$ is regular for any $n\in\mathbb{N}$. Then we have

$$
\begin{aligned}
&vec\left(F_2^{T}P\right)+\left(F_1^{T}\otimes I_{n^2}\right)\sim U_{n\times n^2}vec\left(P_2\right)+\left(F_1^{T}\otimes I_{n^2}\right)\left(D_2^{(n)T}\otimes I_n\right)\sim vec\left(P_2\right) \\
&+vec\left(H_2^{T}QH_1\right)-\left(\left(PG_0R^{-1}G_0^{T}\right)\otimes I_{n^2}\right)\sim U_{n\times n^2}vec\left(P_2\right) \\
&-\left(\left(PG_0R^{-1}G_0^{T}\right)\otimes I_{n^2}\right)\left(D_2^{(n)T}\otimes I_n\right)\sim vec\left(P_2\right)-vec\left(\left(I_n\otimes\left(PG_0^{T}R^{-1}\right)\right)G_1^{T}P\right)=0 \qquad (6.36)
\end{aligned}
$$

Grouping all the terms containing $vec\left(P_2\right)$ in one side and the rest (known terms) in the other side leads to

$$\left[\left(F_1 - G_0 R^{-1} G_0^T P\right)^T \otimes I_{n^2}\right]\left[U_{n\times n^2} + \left(D_2^{(n)T} \otimes I_n\right)\right]\sim vec\left(P_2\right)$$
$$= vec\left[\left(\left(PG_0 R^{-1} G_0^T\right) \otimes I_{n^2}\right)\left(PG_1\right) - H_2^T Q H_1 - F_2^T P\right] \quad (6.37)$$

Since $\left[U_{n\times n^2} + \left(D_2^{(n)T} \otimes I_n\right)\right] = D_3^{(n)T}$ (use the definition of the differential matrix shown in chapter 3), the equation (6.37) will be

$$\left[\left(F_1 - G_0 R^{-1} G_0^T P\right)^T \otimes I_{n^2}\right] D_3^{(n)T} \sim vec\left(P_2\right) = vec\left[\left(-F_2 + G_1\left(I_n \otimes \left(R^{-1} G_0 P\right)\right)\right)^T P - H_2^T Q H_1\right] \quad (6.38)$$

We note

$$\mathscr{F}_2 = \left[\left(F_1 - G_0 R^{-1} G_0^T P\right) \otimes I_{n^2}\right] \quad (6.39)$$

and

$$\mathscr{H}_2 = -vec\left[\left(F_2 - G_1\left(I_n \otimes \left(R^{-1} G_0 P\right)\right)\right)^T P + H_2^T Q H_1\right] \quad (6.40)$$

Hence, the equation (6.38) will be

$$\mathscr{F}_2 D_3^{(n)T} \sim vec\left(P_2\right) = \mathscr{H}_2 \quad (6.41)$$

Since $P = P^T > 0$ is solution of the ARE (6.26), $\left(F_1 - G_0 R^{-1} G_0^T P\right)$ is regular. But, $D_3^{(n)}$ is singular, for any integer $n \geq 2$. We use the non-redundant vector power notation, introduced in chapter 3, to write

$$\tilde{P}_2 = P_2 T_2 \quad (6.42)$$

where $T_2 \in \mathbb{R}^{n^2 \times 2_2^{(n)}}$ with $2_2^{(n)}$ stands for the binomial coefficient [22]. From (6.42), we can write

$$P_2 = \tilde{P}_2 T_2^+ \quad (6.43)$$

where T_2^+ is the Moore-Penrose pseudo-inverse of T_2 defined by

$$T_2^+ = T_2^T \left(T_2 T_2^T\right)^{-1} \tag{6.44}$$

thus, (6.41) becomes

$$\cdot\mathscr{T}_2^T \, D_3^{(n)T} \sim vec\left(\tilde{P}_2 T_2^*\right) = \mathscr{H}_2 \tag{6.45}$$

Using Theorem 3.7, we obtain

$$\sim \cdot\mathscr{T}_2^T \, D_3^{(n)T} \left(T_2^{+T} \otimes I_n\right) vec\left(\tilde{P}_2\right) = \mathscr{H}_2 \tag{6.46}$$

Define

$$T_2 = \left(T_2^+ \otimes I_n\right) D_3^{(n)} \tag{6.47}$$

and T_2^+ its Moore-Penrose pseudo-inverse given by

$$T_2^+ = T_2^T \left(T_2 T_2^T\right)^{-1} \tag{6.48}$$

We obtain from (6.46)

$$vec\left(\tilde{P}_2\right) = \sim^{-1} T_2^{+T} \cdot\mathscr{T}_2^T \mathscr{H}_2 \tag{6.49}$$

then, $P_2 = \tilde{P}_2 T_2^+$ can be deduced.

6.4.3 General order

The calculation of P_p from (6.24) is given by the cancellation of the coefficients of $x^{|p+1|}$. We obtain

$$\underbrace{\sum_{i=1}^{p}\sum_{j=1}^{p}}_{i+j=p+1} vec\left(H_i^T Q H_j\right) + vec\left(V_{p1}^T F_1\right) + vec\left(V_{1p}^T F_1\right) + \underbrace{\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}\sum_{k=1}^{p}}_{i+j+k=p+2} vec\left(V_{ij}^T F_k\right) + vec\left(F_1^T V_{p1}\right)$$

$$+vec\left(F_1^T V_{1p}\right) + \underbrace{\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}\sum_{k=1}^{p}}_{i+j+k=p+2} vec\left(F_k^T V_{ij}\right) - vec\left(W_{p10}^T R^{-1} W_{110}\right) - vec\left(W_{1p0}^T R^{-1} W_{110}\right)$$

$$-vec\left(W_{110}^T R^{-1} W_{p10}\right) - vec\left(W_{110}^T R^{-1} W_{1p0}\right) - \underbrace{\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}\sum_{k=0}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=0}^{p-1}}_{i+j+k+b+c+d=p+3} vec\left(W_{ijk}^T R^{-1} W_{bcd}\right) = 0 \quad (6.50)$$

By replacing V and W by their values according to the definitions (6.19) and (6.23), using (6.12) and applying Theorem 3.9, Theorem 3.10 and the *mat* notation, introduced in chapter 3, the equation (6.50) will be

$$\underbrace{\sum_{i=1}^{p}\sum_{j=1}^{p}}_{i+j=p+1} vec\left(H_i^T Q H_j\right) + \left(F_1^T \otimes I_{n^p}\right) U_{n\times n^p} \sim vec\left(P_p\right) + \left(F_1^T \otimes I_{n^p}\right)\left(D_p^{(n)T} \otimes I_n\right) \sim vec\left(P_p\right)$$

$$+\underbrace{\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}\sum_{k=1}^{p}}_{i+j+k=p+2} vec\left(V_{ij}^T F_h\right) + U_{n^p\times n}\left(F_1^T \otimes I_{n^p}\right) U_{n\times n^p} \sim vec\left(P_p\right)$$

$$+U_{n^p\times n}\left(F_1^T \otimes I_{n^p}\right)\left(D_p^{(n)T} \otimes I_n\right) \sim vec\left(P_p\right) + \underbrace{\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}\sum_{k=1}^{p}}_{i+j+k=p+2} vec\left(F_k^T V_{ij}\right)$$

$$-\left[\left(G_0 R^{-1} G_0^T P\right)^T \otimes I_{n^p}\right] \sim U_{n\times n^p} vec\left(P_p\right) -$$

$$\left[\left(G_0 R^{-1} G_0^T P\right)^T \otimes I_{n^p}\right]\left(D_p^{(n)T} \otimes I_n\right) \sim vec\left(P_p\right)$$

$$-U_{n^p\times n}\left[\left(G_0 R^{-1} G_0^T P\right)^T \otimes I_{n^p}\right] \sim U_{n\times n^p} vec\left(P_p\right)$$

$$-U_{n^p\times n}\left[\left(G_0 R^{-1} G_0^T P\right)^T \otimes I_{n^p}\right]\left(D_p^{(n)T} \otimes I_n\right) \sim vec\left(P_p\right)$$

$$-\underbrace{\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}\sum_{k=0}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=0}^{p-1}}_{i+j+k+b+c+d=p+3} vec\left(W_{ijk}^T R^{-1} W_{bcd}\right) = 0 \quad (6.51)$$

In fact,

$$\begin{aligned} vec\left(V_{p1}^{T}F_{1}\right) &= vec\left[mat_{n^{p}\times n}\left(vec\left(P_{p(1)}^{T}P_{1(p)}D_{1}^{(n)}\right)\right)F_{1}\right] \\ &= vec\left[mat_{n^{p}\times n}\left(vec\left(P_{p}^{T}\sim I_{n}I_{n}\right)\right)F_{1}\right] \\ &= vec\left[mat_{n^{p}\times n}\left(vec\left(P_{p}^{T}\right)\right)\sim F_{1}\right] \\ &= \sim vec\left(P_{p}^{T}F_{1}\right) \\ &= \left(F_{1}^{T}\otimes I_{n^{p}}\right)\sim vec\left(P_{2}^{T}\right) \\ &= \left(F_{1}^{T}\otimes I_{n^{p}}\right)\sim U_{n\times n^{p}}vec\left(P_{p}\right) \end{aligned} \tag{6.52}$$

$$\begin{aligned} vec\left(V_{1p}^{T}F_{1}\right) &= vec\left[mat_{n^{p}\times n}\left(vec\left(P_{1(p)}^{T}P_{p(1)}D_{p}^{(n)}\right)\right)F_{1}\right] \\ &= vec\left[mat_{n^{p}\times n}\left(vec\left(\sim I_{n}P_{p}D_{p}^{(n)}\right)\right)F_{1}\right] \\ &= vec\left[mat_{n^{p}\times n}\left(\sim vec\left(P_{p}D_{p}^{(n)}\right)\right)F_{1}\right] \\ &= \left(F_{1}^{T}\otimes I_{n^{p}}\right)\sim vec\left[mat_{n^{p}\times n}\left(vec\left(P_{p}D_{p}^{(n)}\right)\right)\right] \\ &= \left(F_{1}^{T}\otimes I_{n^{p}}\right)\sim vec\left(P_{p}D_{p}^{(n)}\right) \\ &= \left(F_{1}^{T}\otimes I_{n^{p}}\right)\left(D_{p}^{(n)T}\otimes I_{n}\right)\sim vec\left(P_{p}\right) \end{aligned} \tag{6.53}$$

$$\begin{aligned} vec\left(W_{p10}^{T}R^{-1}W_{110}\right) &= vec\left[mat_{n^{p}\times m}\left(vec\left(V_{p1}^{T}G_{0}\right)\right)R^{-1}\ mat_{n\times m}^{T}\left(vec\left(V_{11}^{T}G_{0}\right)\right)\right] \\ &= vec\left(V_{p1}^{T}G_{0}R^{-1}G_{0}^{T}V_{11}\right) \\ &= vec\left(\sim P_{p}^{T}G_{0}R^{-1}G_{0}^{T}P\right) \\ &= \left(\left(PG_{0}R^{-1}G_{0}^{T}\right)\otimes I_{n^{p}}\right)\sim vec\left(P_{p}^{T}\right) \\ &= \left(\left(PG_{0}R^{-1}G_{0}^{T}\right)\otimes I_{n^{p}}\right)U_{n\times n^{p}}\sim vec\left(P_{p}\right) \\ &= \left(\left(G_{0}R^{-1}G_{0}^{T}P\right)^{T}\otimes I_{n^{p}}\right)U_{n\times n^{p}}\sim vec\left(P_{p}\right) \end{aligned} \tag{6.54}$$

$$vec\left(W_{1p0}^{T}R^{-1}W_{110}\right)=vec\left[mat_{n^p\times m}\left(vec\left(V_{1p}^{T}G_0\right)\right)R^{-1}\ mat_{n\times m}^{T}\left(vec\left(V_{11}^{T}G_0\right)\right)\right]$$
$$=vec\left(V_{1p}^{T}G_0R^{-1}G_0^{T}V_{11}\right)$$
$$=vec\left(V_{1p}^{T}G_0R^{-1}G_0^{T}P\right)$$
$$=\left(\left(PG_0R^{-1}G_0^{T}\right)\otimes I_{n^p}\right)\left(D_p^{(n)T}\otimes I_n\right)\sim\ vec\left(P_p\right)$$
$$=\left(\left(G_0R^{-1}G_0^{T}P\right)^{T}\otimes I_{n^p}\right)\left(D_p^{(n)T}\otimes I_n\right)\sim\ vec\left(P_p\right) \qquad (6.55)$$

Using Theorem 3.9, introduced in chapter 3, we write

$$vec\left(F_1^{T}V_{p1}\right)=U_{n^p\times n}vec\left(V_{p1}^{T}F_1\right)$$
$$=U_{n^p\times n}\left(F_1^{T}\otimes I_{n^p}\right)U_{n\times n^p}\sim\ vec\left(P_p\right) \qquad (6.56)$$

$$vec\left(F_1^{T}V_{1p}\right)=U_{n^p\times n}vec\left(V_{1p}^{T}F_1\right)$$
$$=U_{n^p\times n}\left(F_1^{T}\otimes I_{n^p}\right)\left(D_p^{(n)T}\otimes I_n\right)\sim\ vec\left(P_p\right) \qquad (6.57)$$

$$vec\left(W_{110}^{T}R^{-1}W_{p10}\right)=U_{n^p\times n}vec\left(W_{p10}^{T}R^{-1}W_{110}\right)$$
$$=U_{n^p\times n}\left(\left(PG_0R^{-1}G_0^{T}\right)\otimes I_{n^p}\right)\sim\ U_{n\times n^p}\ vec\left(P_p\right)$$
$$=U_{n^p\times n}\left(\left(G_0R^{-1}G_0^{T}P\right)^{T}\otimes I_{n^p}\right)\sim\ U_{n\times n^p}\ vec\left(P_p\right) \qquad (6.58)$$

$$vec\left(W_{110}^{T}R^{-1}W_{1p0}\right)=U_{n^p\times n}vec\left(W_{1p0}^{T}R^{-1}W_{110}\right)$$
$$=U_{n^p\times n}\left(\left(PG_0R^{-1}G_0^{T}\right)\otimes I_{n^p}\right)\left(D_p^{(n)T}\otimes I_n\right)\sim\ vec\left(P_p\right)$$
$$=U_{n^p\times n}\left(\left(G_0R^{-1}G_0^{T}P\right)^{T}\otimes I_{n^p}\right)\left(D_p^{(n)T}\otimes I_n\right)\sim\ vec\left(P_p\right) \qquad (6.59)$$

Note that, by definition, $D_p^{(n)T}\otimes I_n+U_{n\times n^p}=D_{p+1}^{(n)T}$. In fact,

$$D_p^{(n)T} \otimes I_n + U_{n\times n^p} = \left(\sum_{i=0}^{p-1} U_{n^i\times n} \otimes I_{n^{p-1-i}}\right)^T \otimes I_n + U_{n\times n^p}$$

$$= \left(\sum_{i=0}^{p-1} U_{n^i\times n} \otimes I_{n^{p-1-i}} \otimes I_n + U_{n^p\times n}\right)^T$$

$$= \left(\sum_{i=0}^{p-1} U_{n^i\times n} \otimes I_{n^{p-i}} + U_{n^p\times n}\right)^T$$

$$= \left(\sum_{i=0}^{p-1} U_{n^i\times n} \otimes I_{n^{p-i}} + U_{n^p\times n} \otimes I_{n^0}\right)^T$$

$$= \left(\sum_{i=0}^{p} U_{n^i\times n} \otimes I_{n^{p-i}}\right)^T$$

$$= D_{p+1}^{(n)T} \tag{6.60}$$

If we group all the unknown terms in P_p in one side and all the known terms (the remaining terms) in the other side, the equation (6.51) will be

$$\left(I_{n^{p+1}} + U_{n^p\times n}\right)\left[\left(F_1 - G_0 R^{-1} G_0^T P\right)^T \otimes I_{n^p}\right] D_{p+1}^{(n)T} \sim vec\left(P_p\right) =$$

$$\underbrace{\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}\sum_{k=0}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=0}^{p-1}}_{i+j+k+b+c+d=p+3} vec\left(W_{ijk}^T R^{-1} W_{bcd}\right) - \underbrace{\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}\sum_{k=1}^{p}}_{i+j+k=p+2} vec\left(V_{ij}^T F_k\right)$$

$$- \underbrace{\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}\sum_{k=1}^{p}}_{i+j+k=p+2} vec\left(F_k^T V_{ij}\right) - \underbrace{\sum_{i=1}^{p}\sum_{j=1}^{p}}_{i+j=p+1} vec\left(H_i^T Q H_j\right) \tag{6.61}$$

We note

$$\mathscr{T}_P = \left(F_1 - G_0 R^{-1} G_0^T P\right) \otimes I_{n^p} \tag{6.62}$$

$$\mathscr{H}_P = \underbrace{\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}\sum_{k=0}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=0}^{p-1}}_{i+j+k+b+c+d=p+3} vec\left(W_{ijk}^T R^{-1} W_{bcd}\right) - \underbrace{\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}\sum_{k=1}^{p}}_{i+j+k=p+2} vec\left(V_{ij}^T F_k\right)$$
$$-\underbrace{\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}\sum_{k=1}^{p}}_{i+j+k=p+2} vec\left(F_k^T V_{ij}\right) - \underbrace{\sum_{i=1}^{p}\sum_{j=1}^{p}}_{i+j=p+1} vec\left(H_i^T Q H_j\right) \tag{6.63}$$

Hence, the equation (6.61) will be

$$\left(I_{n^{p+1}} + U_{n^p \times n}\right) \mathscr{F}_P^T D_{p+1}^{(n)T} \tilde{} \; vec\left(P_p\right) = \mathscr{H}_P \tag{6.64}$$

Note that $\mathscr{F}_P^T$ is regular since $\left(F_1 - G_0 R^{-1} G_0^T P\right)$ is a Hurwitz matrix [3]. $D_{p+1}^{(n)T}$ is a singular matrix for all integers $p \geq 2$. $\left(I_{n^{p+1}} + U_{n^p \times n}\right)$ is regular for p even and singular for p odd. Using the non-redundant vector power notation introduced in Definition 3.6 of chapter 3, we can write

$$\tilde{P}_p = P_p T_p \tag{6.65}$$

where $T_p \in \mathbb{R}^{n^p \times 2_p^{(n)}}$, with $2_p^{(n)}$ stands for the binomial coefficient [22]. From (6.65), we can write

$$P_p = \tilde{P}_p T_p^+ \tag{6.66}$$

where T_p^+ is the Moore-Penrose pseudo-inverse of T_p defined by

$$T_p^+ = T_p^T \left(T_p T_p^T\right)^{-1} \tag{6.67}$$

Two cases arise depending on p even or odd.

6.4.3.1 p even

We combine (6.64) and (6.66)

$$\left(I_{n^{p+1}} + U_{n^p \times n}\right) \mathscr{F}_P^T D_{p+1}^{(n)T} \tilde{} \; vec\left(\tilde{P}_p T_p^+\right) = \mathscr{H}_P \tag{6.68}$$

By applying Theorem 3.10, introduced in chapter 3, (6.68) will be

$$\left(I_{n^{p+1}}+U_{n^p\times n}\right)\mathscr{F}_P^{\,T}D_{p+1}^{(n)T}\left(T_p^{+T}\otimes I_n\right)\sim vec\left(\tilde{P}_p\right)=\mathscr{H}_P \tag{6.69}$$

Define

$$T_p=\left(T_p^{+}\otimes I_n\right)D_{p+1}^{(n)} \tag{6.70}$$

Then, (6.69) will be

$$\left(I_{n^{p+1}}+U_{n^p\times n}\right)\mathscr{F}_P^{\,T}T_p^{T}\sim vec\left(\tilde{P}_p\right)=\mathscr{H}_P \tag{6.71}$$

Given T_p^{+} the Moore-Penrose pseudo-inverse of T_p

$$T_p^{+}=T_p^{T}\left(T_p\cdot T_p^{T}\right)^{-1} \tag{6.72}$$

Hence, considering the fact that $\left(I_{n^{p+1}}+U_{n^p\times n}\right)$ is regular for any p even (see Theorem 3.18 in chapter 3), (6.52) will be

$$vec\left(\tilde{P}_p\right)=\sim^{-1}T_p^{+T}\mathscr{F}_P^{-T}\left(I_{n^{p+1}}+U_{n^p\times n}\right)^{-1}\mathscr{H}_P \tag{6.73}$$

If P, P_2, …, P_{p-1} are known, $\tilde{P}_p$ can be calculated from (6.73). Then, $P_p=\tilde{P}_pT_p^{+}$ is deduced.

6.4.3.2 p odd

Note that $\left(I_{n^{p+1}}+U_{n^p\times n}\right)$ is singular for any integer $n\geq 2$ and any integer p odd. The equation (6.24) can be written in terms of $\tilde{x}^{|j|}$. Then, the cancellation of the coefficients of $x^{|p+1|}$ lead to (6.44), but pre-multiplied by T_{p+1}^{T}.

$$T_{p+1}^{T}\left(I_{n^{p+1}}+U_{n^p\times n}\right)\mathscr{F}_P^{\,T}D_{p+1}^{(n)T}\sim vec\left(P_p\right)=T_{p+1}^{T}\mathscr{H}_P \tag{6.74}$$

Define

$$\tilde{\mathscr{H}}_P=T_{p+1}^{T}\mathscr{H}_P \tag{6.75}$$

Then, (6.74) becomes

$$T_{p+1}^{T}\left(I_{n^{p+1}}+U_{n^{p}\times n}\right)\mathcal{F}_P^{\,T}D_{p+1}^{(n)T}\sim vec\left(P_p\right)=\tilde{\mathcal{H}}_P \tag{6.76}$$

Since $P_p=\tilde{P}_pT_p^{+}$, by applying Theorem 3.10, in chapter 3, (6.57) will be

$$T_{p+1}^{T}\left(I_{n^{p+1}}+U_{n^{p}\times n}\right)\mathcal{F}_P^{\,T}D_{p+1}^{(n)T}\left(T_p^{+T}\otimes I_n\right)\sim vec\left(\tilde{P}_p\right)=\tilde{\mathcal{H}}_P \tag{6.77}$$

Using $T_p=\left(T_p^{+}\otimes I_n\right)D_{p+1}^{(n)}$, we obtain

$$T_{p+1}^{T}\left(I_{n^{p+1}}+U_{n^{p}\times n}\right)\mathcal{F}_P^{\,T}T_p^{T}\sim vec\left(\tilde{P}_p\right)=\tilde{\mathcal{H}}_P \tag{6.78}$$

Define

$$\tilde{\mathcal{F}}_P=T_p\mathcal{F}_P\left(I_{n^{p+1}}+U_{n^{p}\times n}\right)T_{p+1} \tag{6.79}$$

and $\tilde{\mathcal{F}}_P^{+}$ its Moore Penrose pseudo-inverse

$$\tilde{\mathcal{F}}_P^{+}=\left(\tilde{\mathcal{F}}_P^{T}\tilde{\mathcal{F}}_P\right)^{-1}\tilde{\mathcal{F}}_P^{T} \tag{6.80}$$

Hence, we obtain

$$vec\left(\tilde{P}_p\right)=\sim^{-1}\tilde{\mathcal{F}}_P^{+T}\tilde{\mathcal{H}}_P \tag{6.81}$$

If P, P_2, …, P_{p-1} are known, $\tilde{P}_p$ can be calculated from (6.81). Then, $P_p=\tilde{P}_pT_p^{+}$ is deduced.

6.5 State feedback design

Consider the nonlinear dynamics (6.1). The optimal control which minimizes the functional cost (6.6) is obtained by the optimality conditions (6.8) and (6.9). We propose to use the procedure introduced in sections 6.2 and 6.3 with an optimal cost $V\left(x\right)$ in the form of (6.10). To solve the obtained nonlinear SDR equation (6.9), transformed in the form of (6.23), it was shown that the cancellation of the terms $x^{|2|}$, $x^{|3|}$, …, $x^{|p+1|}$, ... leads to independent equations in P, P_2, …, P_p,...

respectively. The optimal control can be introduced in an analytical form $u^*(x)$ by using (6.8), (6.19) and (6.22).

$$u^*(x) = -\sum_{p\geq 1} K_p x^{|p|} \tag{6.82}$$

with

$$K_p = R^{-1} \cdot \underbrace{\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=0}^{p-1}}_{i+j+k=p+1} W_{ijk} \tag{6.83}$$

Thus, the KP tensor used here allows a systematic determination of the optimal state-feedback. In practice, we proceed by the design of an approximated suboptimal cost $\bar{V}(x)$ in form (6.10) which leads to the computation of finite number of independent equations in P, P_2, …, $P_{\bar{p}}$. The obtained suboptimal control will be simply truncated at maximum order of $2\bar{p}+g-1$, as follows

$$\bar{u}(x) = \sum_{p=1}^{2\bar{p}+g-1} K_p x^{|p|} \tag{6.84}$$

where g is the order of the polynomial term $G(x)$ introduced in (6.4). Note that the proposed nonlinear feedback (6.83) and (6.84) will not necessarily be implemented with a great number of matrices P_p to be so different from the linear control approximation. It can be concluded that the state-feedback obtained with only P (*i.e.*, only the first order of the SDR equation) is more efficient than the solution issued from the linearized system according to [3]. In fact, by computing only P, we may obtain a polynomial sub-optimal control of order $g+1$ (where g is the order of the term $G(x)$ in (6.1)), in particular, when g is non-zero. The stability of the proposed sub-optimal state feedback (6.83) and (6.84) will be discussed in a further work [57] by considering the approximated cost function $\bar{V}(x)$ as a Lyapunov candidate function.

6.6 Stability discussion

In this section, we discuss thoroughly the stability of the closed loop system in large by considering the analytic expression of $V(x)$ given by (6.10) as a Lyapunov candidate function. In one hand, using (6.10) and (6.11), we write

$$V(x)=\begin{pmatrix} x^T & x^{|2|^T} & \ldots & x^{|j|^T} & \ldots \end{pmatrix}\begin{pmatrix} P & \sim P_2 & \cdots & \sim P_j & \cdots \\ \sim P_2^T & P_2^T P_2 & \cdots & P_2^T P_j & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ \sim P_j^T & P_j^T P_2 & \cdots & P_j^T P_j & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix}\begin{pmatrix} x \\ x^{|2|} \\ \vdots \\ x^{|j|} \\ \vdots \end{pmatrix} \tag{6.85}$$

$P, P_2, ..., P_j, ...$ are calculated using (6.26), (6.73),(6.81), and are independent of $\sim$. If

$$\begin{pmatrix} P & \sim P_2 & \cdots & \sim P_j & \cdots \\ \sim P_2^T & P_2^T P_2 & \cdots & P_2^T P_j & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ \sim P_j^T & P_j^T P_2 & \cdots & P_j^T P_j & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} > 0 \tag{6.86}$$

then, $V(x) > 0, \forall x \neq 0$. Note that from (6.10) we obtain $V(x) > 0$ if $P = P^T > \sim^2 I$. In the other hand, the time derivative of $V(x)$ along the trajectories of the closed loop system (6.1) with the optimal control $u^*(x)$, given by (6.82), is [5]

$$\begin{aligned}
\dot{V}(t) &= \frac{\partial V^T}{\partial x} \cdot \dot{x}(t) \\
&= V_x(x)^T \cdot \left(F(x) - G(x) R^{-1} G^T(x) V_x(x)\right) \\
&= -\frac{1}{2} H(x)^T Q H(x) - \frac{1}{2} V_x(x)^T G(x) R^{-1} G^T(x) V_x(x) \\
&= -\frac{1}{2} H(x)^T Q H(x) - \frac{1}{2} u^*(x)^T R u^*(x)
\end{aligned} \tag{6.87}$$

Noting that $Q = Q^T \geq 0$ and $R = R^T > 0$, then $\dot{V}(t) < 0, \forall x \neq 0$ [5]. Consequently, if $P > \sim^2 I$ holds, then the optimal state-feedback control (6.1)-(6.9) is GAS. The stability analysis of the closed-loop sub-optimal control is discussed in details in [57] in terms of LMI feasibility problems including the estimation of the domain of attraction of the designed control. Such LMI problems can be solved numerically by using any interior point optimization method implemented in MATLAB using the LMI control toolbox (see [57] and references cited therein).

6.7 Numerical applications

Example 6.1 (Scalar Example): Consider the following dynamic model (same as Example 5.1 of chapter 5) defined by the dynamics

$$\dot{x} = -1.4x - 0.5x^2 + (0.5 - 2.5x)\cdot u \tag{6.88}$$

where $x \in \mathbb{R}$ and $u \in \mathbb{R}$ are the state and the control input, respectively. We recall the "exact" optimal controller and the 3^{rd} order KP-based controller

$$u_{exact} = \frac{7x + 2.5x^2 - x\sqrt{156.25x^2 - 25x + 55}}{12.5(0.2 - x)} \tag{6.89}$$

$$u_{KP} = -0.1664x + 0.8418x^2 + 0.0719x^3 \tag{6.90}$$

By applying the algorithm proposed in this chapter using (6.12), (6.19), (6.23), (6.26), (6.73), (6.81), (6.83) and (6.84), we design a new Lyapunov-based KP controller. The LF based controller of 3^{rd} order is given by

$$u_{LF} = -0.1664x + 0.8355x^2 + 0.1018x^3 \tag{6.91}$$

Obviously, we can see that the linear controller designed from the linearized system is

$$u_{Lin} = -0.1664x \tag{6.92}$$

The design and the simulation of the proposed four (Exact, Linear, KP and LF) techniques lead to the suboptimal costs J_{exact}, J_{Lin}, J_{KP} and J_{LF}. The results are presented in Table 6.1 for different initial conditions, $x(0)$, in terms of the cost value J_{exact} for the exact design, and the relative cost errors in %, $£_{Lin} = \left|\frac{J_{Lin} - J_{Exact}}{J_{Exact}}\right|$ for the Linear design, $£_{KP(n)} = \left|\frac{J_{KP} - J_{Exact}}{J_{Exact}}\right|$ for the n^{th} KP design and $\pounds_{LF(n)} = \left| \frac{J_{LF} - J_{Exact}}{J_{Exact}} \right|$ for the n^{th} LF design. The selected truncation orders are $n = 2$ and $n = 3$ respectively.

Table 6.1 Exact cost and Sub-optimal costs Errors *vs.* Initial condition for the scalar example (6.88)

$x(0)$	J_{exact}	$\pounds_{Lin}$	$\pounds_{KP(2)}$	$\pounds_{LF(2)}$	$\pounds_{KP(3)}$	$\pounds_{LF(3)}$
1.0	0.3163	26.84	1.45	**0.13**	2.37	**0.03**
2.0	0.9112	70.01	13.28	**1.02**	20.02	**0.02**
3.0	1.5857	112.52	32.55	**2.19**	49.37	**0.32**
4.0	2.2954	152.68	55.07	**3.37**	85.40	**1.49**
5.0	3.0250	190.40	79.14	**4.47**	126.17	**2.10**
6.0	3.7673	225.85	104.06	**5.49**	170.89	**1.73**
7.0	4.5184	259.23	129.48	**6.43**	215.19	**1.36**
8.0	5.2759	290.74	155.21	**7.29**	270.84	**1.32**
9.0	6.0385	320.53	181.14	**8.09**	325.74	**1.38**
10.0	6.8049	348.77	207.22	**8.84**	383.83	**1.64**

The results of the different simulations show that the cost errors are much lower with the LF-KP design than the other methods (Linear and KP), (see $\pounds_{LF(2)}$ and $\pounds_{LF(3)}$ Columns in Table 6.1). Furthermore, for the performance of the 3^{rd} order LF design is better than those of the second order. The errors decrease and we get a satisfactory improvement in terms of cost estimation, curve fitting of the state variable *w.r.t.* to the "exact" solution and input magnitude. This improvement is supported also by Figures 6.1 and 6.2 which show the process variable x and the input variable u evolutions for an initial condition $x(0) = 6.0$ with the different techniques of control ("exact" design, Linear design, KP design of $n = 2$ *and* $n = 3$ and LF design of $n = 2$ *and* $n = 3$).

The simulation results for state evolution shows that best fitting *w.r.t.* the exact controller is given by the 3^{rd} order LF one, which presents the best predicted behaviour of the exact controller.

In terms of input control, the best fitting is guaranteed by the LF 2nd order controller, presenting almost the same behaviour as the exact controller.

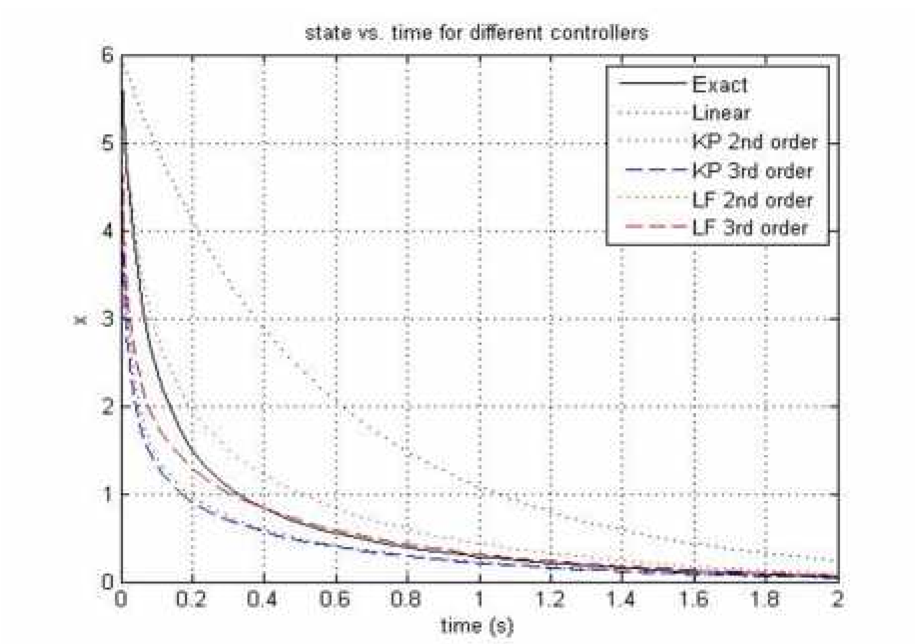

Fig 6.1 State evolution for different controllers of the scalar example (6.88)

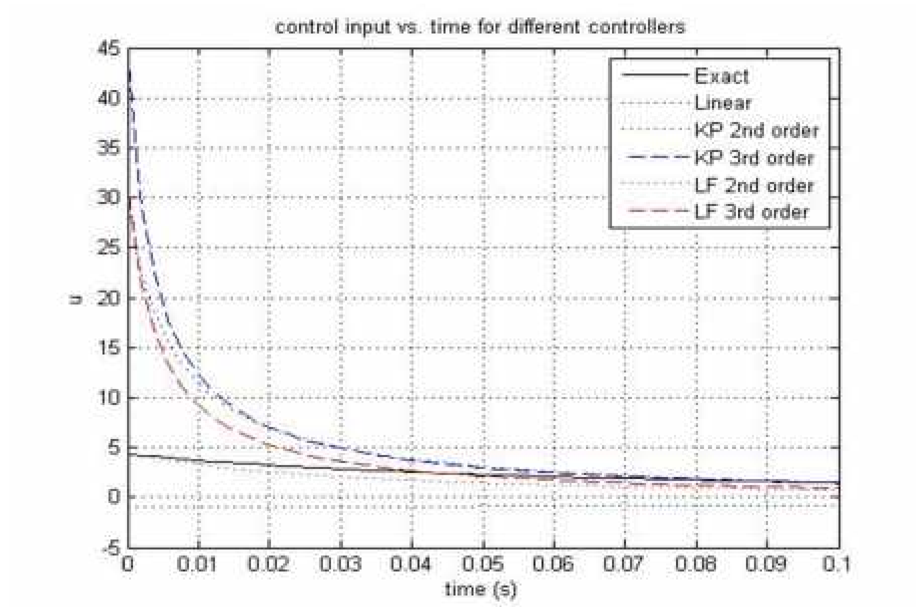

Fig 6.2 Input control evolution for different controllers of the scalar example (6.88)

Example 6.2 (F8 Fighter Model): Consider the F8 fighter dynamics model [59]

$$\dot{x}_1 = -0.88x_1 + x_3 - 0.09x_1x_3 + 0.47x_1^2 - 0.02x_2^2 - x_1^2x_3 + 3.85x_1^3$$
$$-0.21u + 0.28x_1^2u + 0.47x_1u^2 + 0.63u^3 \quad (6.93)$$

$$\dot{x}_2 = x_3 \quad (6.94)$$

$$\dot{x}_3 = -4.21x_1 - 0.40x_3 - 0.47x_1^2 - 3.56x_1^3 - 20.97u + 6.26x_1^2u$$
$$+46x_1u^2 + 61.40u^3 \quad (6.95)$$

with the optimal cost function

$$J = \frac{1}{2}\int_0^\infty \left(x^TQx + u^TRu\right)dt \quad (6.96)$$

where $x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T$ is the state vector, $Q = diag\begin{pmatrix} 0.25, & 0.25, & 0.25 \end{pmatrix}$ and $R = 1$. Note that the terms involving nonlinearities in u with small effect on the dynamics are eliminated, as the approaches discussed here cannot account for nonlinear control terms [59], but are taken into consideration in the simulated model dynamics. The simulations have been applied with the proposed LF-based technique as well as the linear control, Lin, where the model is linearized about the origin, the KP-based design introduced in [3] and an SDR-equation-pointwise-based technique [59] (referred to as pw in the following). The sub-optimal cost $J_{\bullet}$ is evaluated with different initial conditions in terms of angle of attack, that is $x_1(0)$ in degree, but with the same initial conditions $x_2(0)=x_3(0)=0$, for the different methods. Table 6.2 shows the cost performance errors $\pounds_J^{\bullet}=\dfrac{J_{\text{pw}}-J_{\bullet}}{J_{\text{pw}}}$ in %; LF- (with $p=2$ and $p=3$), KP- (with $p=2$ and $p=3$) and Lin-based design costs are compared to the pw-technique one. A positive value corresponds to an improvement (*i.e.*, a lower cost) with the given method compared to the pw one; meanwhile the negative value means a higher cost. The LF design discussed in this chapter exhibits the best results in terms of cost performance.

Table 6.2 Cost index J^{pw} and Sub-optimal costs errors *vs.* initial conditions for the F8 fighter

$x_1(0)$	J^{pw}	$\pounds_{J(p=2)}^{\text{LF}}$	$\pounds_{J(p=3)}^{\text{LF}}$	$\pounds_{J(p=2)}^{\text{KP}}$	$\pounds_{J(p=3)}^{\text{KP}}$	$\pounds_{J}^{\text{Lin}}$
6°	0.0016	**<u>20.2</u>**	**18.6**	–0.6	–0.8	0.0
12°	0.0071	**<u>23.8</u>**	**22.8**	–1.6	–2.6	–0.2
17°	0.0196	**<u>30.9</u>**	**30.3**	–3.7	–6.8	–0.7
23°	0.0519	**<u>46.3</u>**	**45.7**	–13.3	–31.7	–4.3
29°	0.1056	**<u>48.3</u>**	**46.3**	Unstable	Unstable	Unstable
34°	0.4081	**<u>71.4</u>**	**65.6**	Unstable	Unstable	Unstable
40°	1.6170	**<u>58.5</u>**	**50.9**	Unstable	Unstable	Unstable

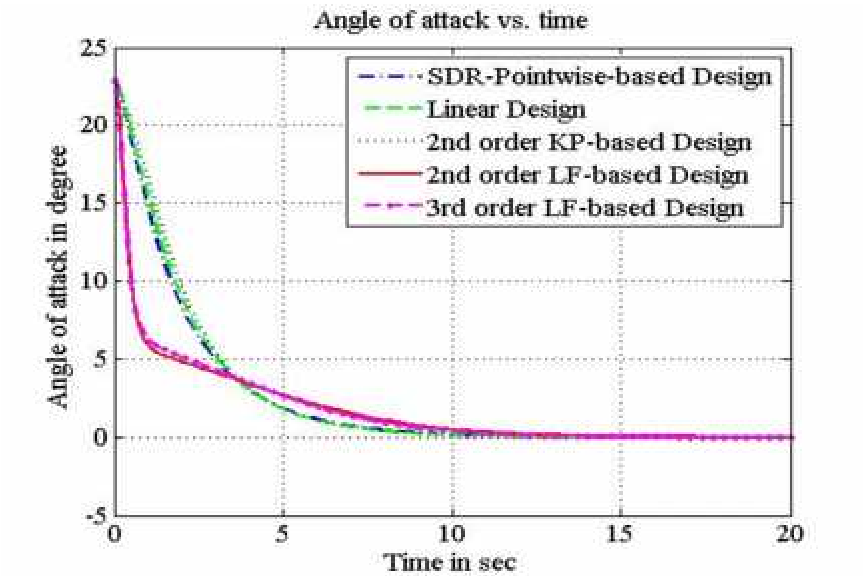

Figure. 6.3 Angle of attack evolution for different controllers of the F8 fighter

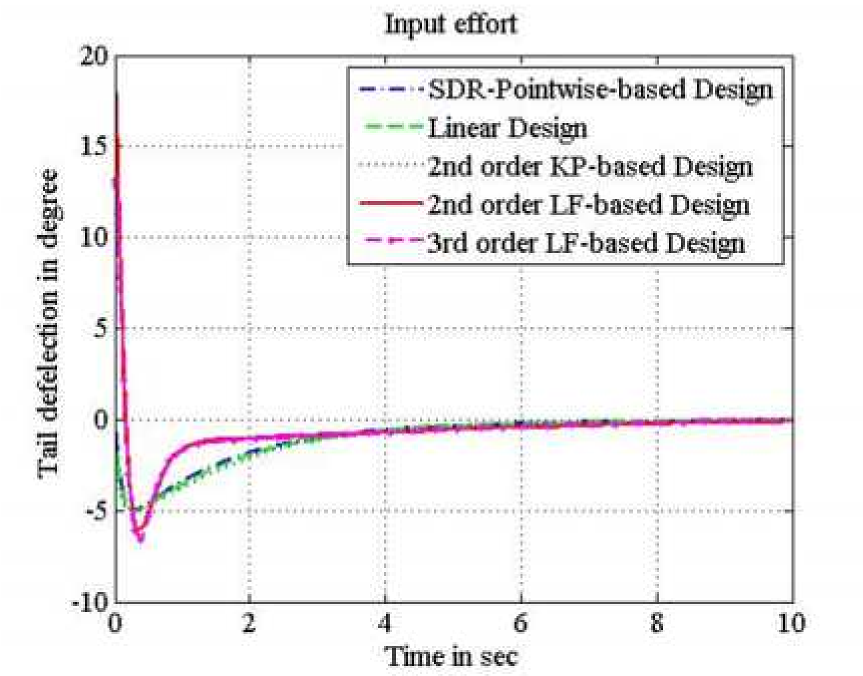

Figure. 6.4. Input control evolution for different controllers of the F8 fighter

Figures 6.3 and 6.4 show the angle of attack and the input control simulation results, obtained with the initial condition $x_1(0) = 23^\circ$. Simulations of LF-based design, with order of truncation of $p = 2$ and $p = 3$, overlap almost during all the time. They show very similar results in terms of transient behaviour and stability. Furthermore, the proposed LF design (with both orders $p = 2$ and $p = 3$ which remain relatively small) exhibits a significant added-value in terms of cost estimation and domain of attraction interval performances compared to the other methods.

6.8 Conclusion

In this chapter, we presented the method of optimal control using the KP-LF-based method. After introducing this chapter in section 6.1, we stated in section 6.2 the problem of optimal control. The main contributions in terms of stability framework is shown in section 6.3, in which we presented the equations of approximations which transform the main equation to uncoupled linear equations and by choosing the cost function in a quadratic form satisfying the Lyapunov candidate function conditions to guarantee the GAS. The resolution of the algorithms of theses equations was presented in section 6.4, in which we calculated the different gain matrices for different orders of truncation using the KP algebra. In section 6.5, we presented the corresponding state feedback optimal control law. In section 6.6, we checked roughly the stability of the closed loop system. In chapter 6.7, we illustrated the improvement in terms of control performance of the new technique through two nonlinear plants. Finally, in section 6.8, we conclude this chapter.

7 Application to a 2-DOF helicopter model based setup – simulations and experiments

7.1 Introduction

The main objective of this chapter is to apply the proposed design method to a 2-DOF helicopter-model-based setup in order to test its efficiency. In the first section, we introduce this chapter. In section 7.2, we give a brief description of the system. In the third section, we present the model dynamics. Then, in section 7.4, we present the design method of the proposed linear and nonlinear controllers of different orders of truncations. In section 7.5, we present the simulation results of the proposed controllers. In section 7.6, we present the experimental results for the same controllers. Finally, we conclude this chapter in section 7.7. Note that further simulations and experimental results have been completed in Appendices E and F respectively for different desired trajectories.

7.2 Description of the system

The 2-DOF helicopter model, as shown in Figure 7.1 and designed by Quanser Inc, consists of a helicopter model mounted on a fixed base with two propellers that are driven by DC motors. The front propeller controls the elevation of the helicopter nose about the pitch axis and the back propeller controls the side to side motions of the helicopter about the yaw axis. The pitch and yaw angles are measured using high resolution encoders [61]. The helicopter has two DC motors: the yaw motor, actuating the back propeller and pitch motor rotating the front propeller.

Figure 7.1 Quanser 2-DOF helicopter set up

The helicopter-based setup has two encoders measuring the pitch angle and the yaw angle [61]. The yaw motor has an armature resistance of 1.6Ω and a current torque constant of $0.0109\,Nm/A$. The larger pitch motor has an armature resistance of 0.83Ω and a current torque constant of $0.0182\,Nm/A$. The pitch motor/propeller has an identified thrust force constant of $1.04\,N/V$ and the yaw motor/propeller has a thrust force constant of $0.43\,N/V$. Table 7.1 summarizes the main electrical and mechanical proprieties of the 2-DOF helicopter system, as depicted from [61].

Table 7.1 Electrical and Mechanical proprieties of the 2-DOF set up [61]

Symbol	**Description**	**Value**	**Unit**
$J_{eq,p}$	Total moment of inertia about pitch axis	0.0384	$Kg.m^2$
$J_{eq,y}$	Total moment of inertia about pitch axis	0.0432	$Kg.m^2$
m_{hel}	Total moving mass of helicopter	1.3872	Kg
l_{cm}	Center of mass length along helicopter body from pitch axis	0.0186	M
B_p	Equivalent viscous damping about pitch axis	0.800	N/V
B_y	Equivalent viscous damping about yaw axis	0.318	N/V
K_{pp}	Thrust torque constant acting on pitch axis from pitch /propeller	0.204	N.m/V
K_{py}	Thrust torque constant acting on pitch axis from yaw /propeller	0.0068	N.m/V
K_{yy}	Thrust torque constant acting on yaw axis from yaw /propeller	0.072	N.m/V
K_{yp}	Thrust torque constant acting on yaw axis from pitch /propeller	0.0219	N.m/V

7.3 Dynamics of the system

7.3.1 Model of the 2-DOF helicopter and state space representation

The 2-DOF helicopter pivots about the pitch axis by angle ‾ and about the yaw axis by angle · . As shown in Figure 7.2, the pitch is defined positive when the nose of the helicopter goes up and the yaw is defined positive for a clockwise rotation. Figure 7.2 shows the thrust force F_p acting on the pitch axis that is normal to the plane of the front propeller and a thrust force F_y acting on the yaw axis that is normal to the rear propeller. Therefore a pitch torque is being applied at a distance r_p from the pitch axis and a yaw torque is applied at a distance r_y from the yaw axis. The gravitational force F_g generates a torque at the helicopter center of mass that pulls down on the helicopter nose. As shown in Figure 7.2, the center of mass is at a distance of l_{cm} from the pitch axis along the helicopter body length

[61]. The body center of mass is to be described in xyz cartesian coordinates *w.r.t.* the pitch θ and yaw ψ angles.

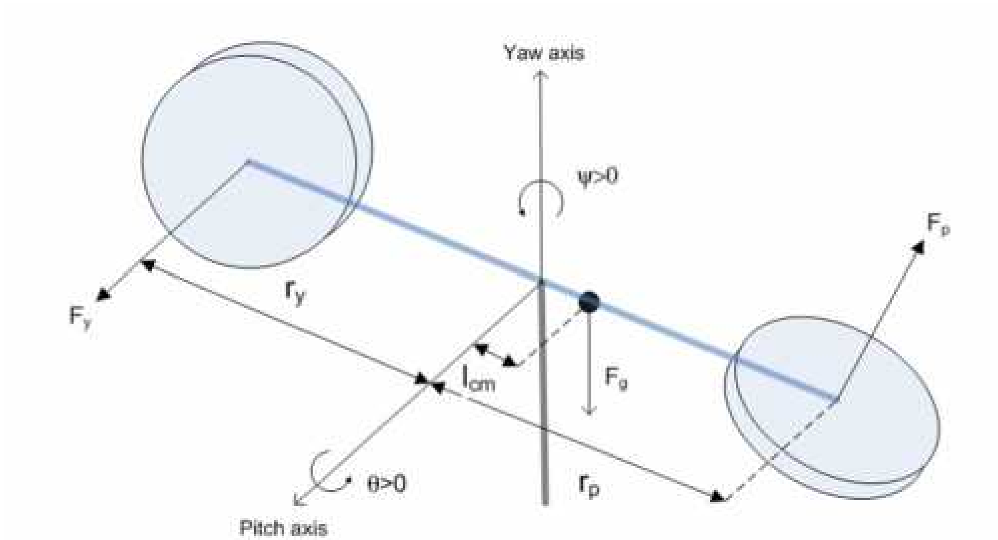

Figure 7.2 Simplified free body diagram of the 2-DOF helicopter [61]

The equation of motion of this system is given by [61]

$$\begin{cases} \left(J_{eq,p} + m_{hel} l_{cm}^2\right)\ddot{\theta} = -m_{hel} g l_{cm} \cos\theta - B_p \dot{\theta} - m_{hel} l_{cm}^2 \sin\theta \cos\theta \dot{\psi}^2 + K_{pp} V_{m,p} + K_{py} V_{m,y} \\ \left(J_{eq,y} + m_{hel} l_{cm}^2 \cos\theta^2\right)\ddot{\psi} = 2 m_{hel} l_{cm}^2 \sin\theta \cos\theta \dot{\psi}\dot{\theta} - B_y \dot{\psi} + K_{yp} V_{m,p} + K_{yy} V_{m,y} \end{cases} \quad (7.1)$$

For more details about the modelization of this setup refer to appendix A. We denote by X the state vector

$$X = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \end{bmatrix}^T = \begin{bmatrix} \theta & \psi & \dot{\theta} & \dot{\psi} \end{bmatrix}^T \quad (7.2)$$

We write from (7.2)

$$\begin{cases} \dot{X}_1 = X_3 \\ \dot{X}_2 = X_4 \end{cases} \quad (7.3)$$

We denote by U the input vector

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}^T = \begin{bmatrix} U_{m,p} & U_{m,y} \end{bmatrix}^T \tag{7.4}$$

The equations (7.1)-(7.4) lead to the state space representation

$$\begin{cases} \dot{X}_1 = X_3 \\ \dot{X}_2 = X_4 \\ \dot{X}_3 = -\dfrac{m_{hel} g l_{cm}}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} \cos X_1 - \dfrac{B_p}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} X_3 - \dfrac{m_{hel} l_{cm}^2}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} \sin X_1 \cdot \cos X_1 \cdot X_4^2 + \dfrac{K_{pp}}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} U_1 + \dfrac{K_{py}}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} U_2 \\ \dot{X}_4 = \dfrac{2 m_{hel} l_{cm}^2}{\left(J_{eq,y} + m_{hel} l_{cm}^2 \cos X_1^2\right)} \sin X_1 \cdot \cos X_1 \cdot X_3 \cdot X_4 - \dfrac{B_y}{\left(J_{eq,y} + m_{hel} l_{cm}^2 \cos X_1^2\right)} X_4 + \dfrac{K_{yp}}{\left(J_{eq,y} + m_{hel} l_{cm}^2 \cos X_1^2\right)} U_1 + \dfrac{K_{yy}}{\left(J_{eq,y} + m_{hel} l_{cm}^2 \cos X_1^2\right)} U_2 \end{cases} \tag{7.5}$$

The system (7.5) represents the nonlinear dynamics of the 2-DOF helicopter and can be written in compact form as

$$\dot{X} = F(X) + G(X)U = \begin{pmatrix} f_1(X) \\ f_2(X) \\ f_3(X) \\ f_4(X) \end{pmatrix} + \begin{pmatrix} g_{11}(X) & g_{12}(X) \\ g_{21}(X) & g_{22}(X) \\ g_{31}(X) & g_{32}(X) \\ g_{41}(X) & g_{42}(X) \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \tag{7.6}$$

where

$$\begin{cases} f_1(X) = X_3 \\ f_2(X) = X_4 \\ f_3(X) = -\dfrac{m_{hel} g l_{cm}}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} \cos X_1 - \dfrac{B_p}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} X_3 - \dfrac{m_{hel} l_{cm}^2}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} \sin X_1 \cdot \cos X_1 \cdot X_4^{\,2} \\ f_4(X) = \dfrac{2 m_{hel} l_{cm}^2}{\left(J_{eq,y} + m_{hel} l_{cm}^2 \cos x_1^2\right)} \sin X_1 \cdot \cos X_1 \cdot X_3 \cdot X_4 - \dfrac{B_y}{\left(J_{eq,y} + m_{hel} l_{cm}^2 \cos X_1^{\,2}\right)} X_4 \end{cases} \tag{7.7}$$

$$\begin{cases} g_{11}(X) = 0 \\ g_{21}(X) = 0 \\ g_{31}(X) = \dfrac{K_{pp}}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} \\ g_{41}(X) = \dfrac{K_{yp}}{\left(J_{eq,y} + m_{hel} l_{cm}^2 \cos X_1{}^2\right)} \end{cases} \tag{7.8}$$

and

$$\begin{cases} g_{12}(X) = 0 \\ g_{22}(X) = 0 \\ g_{32}(X) = \dfrac{K_{py}}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} \\ g_{42}(X) = \dfrac{K_{yy}}{\left(J_{eq,y} + m_{hel} l_{cm}^2 \cos X_1{}^2\right)} \end{cases} \tag{7.9}$$

7.3.2 Equilibrium

We denote by $X_0 = \begin{bmatrix} X_{10} & X_{20} & X_{30} & X_{40} \end{bmatrix}^T$ and $U = \begin{bmatrix} U_{10} & U_{20} \end{bmatrix}^T$ the state and the input at the equilibrium. We consider at the equilibrium all the states are equal to zero, *i.e.*, $X_{10} = 0$, $X_{20} = 0$, $X_{30} = 0$ and $X_{40} = 0$. Then, from (7.5), we obtain at the equilibrium

$$\begin{cases} K_{pp} U_{10} + K_{py} U_{20} - m_{hel} g l_{cm} = 0 \\ K_{yp} U_{10} + K_{yy} U_{20} = 0 \end{cases} \tag{7.10}$$

By solving (7.10) in U_{10} and U_{20}, we obtain

$$\begin{cases} U_{10} = \dfrac{m_{hel} g l_{cm} K_{yy}}{K_{pp}K_{yy} - K_{py}K_{yp}} \\ U_{20} = -\dfrac{K_{yp}}{K_{yy}} U_{10} = \dfrac{m_{hel} g l_{cm} K_{yp}}{K_{py}K_{yp} - K_{pp}K_{yy}} \end{cases} \tag{7.11}$$

7.3.3 Linearized model and high order approximations

Given the equilibrium point $\left(X_0, U_0\right)$ defined by

$$X_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } U_0 = \begin{pmatrix} \dfrac{m_{hel} g l_{cm} K_{yy}}{K_{pp}K_{yy} - K_{py}K_{yp}} \\ \dfrac{m_{hel} g l_{cm} K_{yp}}{K_{py}K_{yp} - K_{pp}K_{yy}} \end{pmatrix} \tag{7.12}$$

The linearized form (*i.e.*, 1^{st} order approximation) of the dynamics (7.7) about the equilibrium $\left(X_0, U_0\right)$ is given by

$$\begin{cases} \dot{X}_1 = X_3 \\ \dot{X}_2 = X_4 \\ \dot{X}_3 = \dfrac{-B_p}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} X_3 + \dfrac{K_{pp}}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} \left(U_1 - U_{10}\right) + \dfrac{K_{py}}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} \left(U_2 - U_{20}\right) \\ \dot{X}_4 = \dfrac{-B_y}{\left(J_{eq,y} + m_{hel} l_{cm}^2\right)} X_4 + \dfrac{K_{yp}}{\left(J_{eq,y} + m_{hel} l_{cm}^2\right)} \left(U_1 - U_{10}\right) + \dfrac{K_{yy}}{\left(J_{eq,y} + m_{hel} l_{cm}^2\right)} \left(U_2 - U_{20}\right) \end{cases} \tag{7.13}$$

which can be represented in matrix form

$$\dot{x} = F_1 x + G_0 u \tag{7.14}$$

with $x = X$ and $u = U - U_0$ the change *w.r.t.* the equilibrium. The matrices $F_1 = \begin{bmatrix} 0_{2,4} & F_{34} \end{bmatrix}$, F_{34} and G_0 are given by

$$F_{34}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{-B_p}{\left(J_{eq,p}+m_{hel}l_{cm}^2\right)} & 0 \\ 0 & \dfrac{-B_y}{\left(J_{eq,y}+m_{hel}l_{cm}^2\right)} \end{pmatrix},\ G_0=\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{K_{pp}}{\left(J_{eq,p}+m_{hel}l_{cm}^2\right)} & \dfrac{K_{py}}{\left(J_{eq,p}+m_{hel}l_{cm}^2\right)} \\ \dfrac{K_{yp}}{\left(J_{eq,y}+m_{hel}l_{cm}^2\right)} & \dfrac{K_{yy}}{\left(J_{eq,y}+m_{hel}l_{cm}^2\right)} \end{pmatrix} \quad (7.15)$$

The second order approximation of the dynamics (7.7) can be written using the Taylor vector expansion (see sub-section 3.4.1 in chapter 3) about $\left(X_0,U_0\right)$. We obtain

$$\begin{cases} \dot{X}_1=X_3 \\ \dot{X}_2=X_4 \\ \dot{X}_3=\dfrac{-B_p}{\left(J_{eq,p}+m_{hel}l_{cm}^2\right)}X_3+\dfrac{1}{2}\dfrac{m_{hel}gl_{cm}}{\left(J_{eq,p}+m_{hel}l_{cm}^2\right)}X_1^2+\dfrac{K_{pp}}{\left(J_{eq,p}+m_{hel}l_{cm}^2\right)}\left(U_1-U_{10}\right)+\dfrac{K_{py}}{\left(J_{eq,p}+m_{hel}l_{cm}^2\right)}\left(U_2-U_{20}\right) \\ \dot{X}_4=\dfrac{-B_y}{\left(J_{eq,y}+m_{hel}l_{cm}^2\right)}X_4+\dfrac{K_{yp}}{\left(J_{eq,y}+m_{hel}l_{cm}^2\right)}\left(U_1-U_{10}\right)+\dfrac{K_{yy}}{\left(J_{eq,y}+m_{hel}l_{cm}^2\right)}\left(U_2-U_{20}\right) \end{cases} \quad (7.16)$$

which can be represented in matrix form

$$\dot{x}=F_1x+F_2x^{|2|}+G_0u \quad (7.17)$$

where F_1 and G_0 are given by (7.15) and F_2 is built from $F_2=0_{4\times16}$ and $F_2\left(3,1\right)=\dfrac{1}{2}\dfrac{m_{hel}gl_{cm}}{\left(J_{eq,p}+m_{hel}l_{cm}^2\right)}$.

The third order approximation of the dynamics (7.7) can be written using the Taylor vector expansion about $\left(X_0,U_0\right)$. We obtain

$$\begin{cases} \dot{X}_1=X_3 \\ \dot{X}_2=X_4 \\ \dot{X}_3=\dfrac{-B_p}{\left(J_{eq,p}+m_{hel}l_{cm}^2\right)}X_3+\dfrac{1}{2}\dfrac{m_{hel}gl_{cm}}{\left(J_{eq,p}+m_{hel}l_{cm}^2\right)}X_1^2+\dfrac{-m_{hel}l_{cm}^2}{\left(J_{eq,p}+m_{hel}l_{cm}^2\right)}X_1X_4^2+\dfrac{K_{pp}}{\left(J_{eq,p}+m_{hel}l_{cm}^2\right)}\left(U_1-U_{10}\right)+\dfrac{K_{py}}{\left(J_{eq,p}+m_{hel}l_{cm}^2\right)}\left(U_2-U_{20}\right) \\ \dot{X}_4=\dfrac{-B_y}{\left(J_{eq,y}+m_{hel}l_{cm}^2\right)}X_4+\dfrac{2m_{hel}l_{cm}^2}{\left(J_{eq,y}+m_{hel}l_{cm}^2\right)}X_1X_3X_4+\dfrac{K_{yp}}{\left(J_{eq,y}+m_{hel}l_{cm}^2\right)}\left(U_1-U_{10}\right)+\dfrac{K_{yy}}{\left(J_{eq,y}+m_{hel}l_{cm}^2\right)}\left(U_2-U_{20}\right) \end{cases} \quad (7.18)$$

which can be represented in matrix form

$$\dot{x} = F_1 x + F_2 x^{|2|} + F_3 x^{|3|} + G_0 u \tag{7.19}$$

where F_1 and G_0 are given by (7.15) , F_2 is built from $F_2 = 0_{4\times16}$ and $F_2(3,1) = \frac{1}{2}\frac{m_{hel} g l_{cm}}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)}$, and F_3 is built from $F_3 = 0_{4\times4^3}$, $F_3(3,16) = \frac{-m_{hel} l_{cm}^2}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)}$ and $F_3(4,12) = \frac{2 m_{hel} l_{cm}^2}{\left(J_{eq,y} + m_{hel} l_{cm}^2\right)}$.

The fourth order approximation of the dynamics (7.7) can be written using the Taylor vector expansion about $\left(X_0, U_0\right)$. We obtain

$$\begin{cases} \dot{X}_1 = X_3 \\ \dot{X}_2 = X_4 \\ \dot{X}_3 = \frac{-B_p}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} X_3 + \frac{1}{2}\frac{m_{hel} g l_{cm}}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} X_1^2 - \frac{m_{hel} l_{cm}^2}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} X_1 X_4^2 - \frac{1}{24}\frac{m_{hel} g l_{cm}}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)} X_1^4 \\ \qquad + \frac{K_{pp}}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)}\left(U_1 - U_{10}\right) + \frac{K_{py}}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)}\left(U_2 - U_{20}\right) \\ \dot{X}_4 = \frac{-B_y}{\left(J_{eq,y} + m_{hel} l_{cm}^2\right)} X_4 + \frac{2 m_{hel} l_{cm}^2}{\left(J_{eq,y} + m_{hel} l_{cm}^2\right)} X_1 X_3 X_4 + \frac{1}{2}\frac{m_{hel} l_{cm}^2}{\left(J_{eq,y} + m_{hel} l_{cm}^2\right)}\left(K_{yp} U_{10} + K_{yy} U_{20}\right) X_1^4 \end{cases} \tag{7.20}$$

which can represented in matrix form

$$\dot{x} = F_1 x + F_2 x^{|2|} + F_3 x^{|3|} + F_4 x^{|4|} + G_0 u \tag{7.21}$$

where F_1 and G_0 are given by (7.15) , F_2 is built from $F_2 = 0_{4\times16}$ and $F_2(3,1) = \frac{1}{2}\frac{m_{hel} g l_{cm}}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)}$, F_3 is built from $F_3 = 0_{4\times4^3}$,

$F_3(3,16) = \dfrac{-m_{hel} l_{cm}^2}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)}$ and $F_3(4,12) = \dfrac{2 m_{hel} l_{cm}^2}{\left(J_{eq,y} + m_{hel} l_{cm}^2\right)}$, and F_4 is built from

$F_4 = 0_{4\times 4^4}$, $F_4(3,1) = -\dfrac{1}{24}\dfrac{m_{hel} g l_{cm}}{\left(J_{eq,p} + m_{hel} l_{cm}^2\right)}$ and

$F_4(4,1) = \dfrac{1}{2}\dfrac{m_{hel} l_{cm}^2}{\left(J_{eq,y} + m_{hel} l_{cm}^2\right)}\left(K_{yp} U_{10} + K_{yy} U_{20}\right)$.

7.4 Control design

We show in chapter 6 that for a given order of truncation p, the suboptimal control law $\bar{u}$ is given by the equation (6.84). Hence, for the dynamics (7.21), the control law is in the form

$$u = -\left(K_1 x + K_2 x^{|2|} + K_3 x^{|3|} + K_4 x^{|4|}\right) \tag{7.22}$$

the gain matrices K_p, $1 \le p \le 4$, are computed from (6.83). Their numerical values are:

$K_1 = \begin{pmatrix} 14.0 & 2.1 & 7.2 & 1.3 \\ -2.1 & 14.0 & -0.2 & 5.9 \end{pmatrix}$, $K_2 = \begin{pmatrix} 2.4 & -0.4 & 0.2 & -0.1 & 0_1 \\ -0.2 & 0.0 & -0.0 & 0.0 & 0_1 \end{pmatrix}$, $(K_3)_{1-4} = \begin{pmatrix} 385.1 & -187.0 & -2.3 & -44.5 \\ 6.4 & -7.7 & -12.1 & -2.1 \end{pmatrix}$, *i.e.* the columns 1 to 4 of K_3, $(K_3)_{5-8} = \begin{pmatrix} -77.4 & 32.1 & 14.7 & 9.8 \\ -1.2 & 1.3 & 1.8 & 0.4 \end{pmatrix}$, *i.e.* the columns 5 to 8 of K_3, $(K_3)_{5-1} = \begin{pmatrix} -6.2 & 3.7 & -3.7 & 0.5 \\ 1.1 & 0.9 & 0.3 & 0.4 \end{pmatrix}$, *i.e.* the columns 9 to 12 of K_3, $(K_3)_{1\ -1} = \begin{pmatrix} -16.3 & 9.7 & 4.0 & 3.2 \\ -0.3 & 0.4 & 0.3 & 0.1 \end{pmatrix}$, *i.e.* the columns 13 to 16 of K_3, $(K_3)_{1\ -2} = \begin{pmatrix} -52.4 & 24.9 & 14.9 & 8.2 \\ -4.1 & 1.3 & 2.0 & 0.4 \end{pmatrix}$, *i.e.* the columns 17 to 20 of K_3, $(K_3)_{2\ -2} = \begin{pmatrix} -11.5 & 7.1 & -11.9 & 1.7 \\ 0.7 & 0.5 & 0.7 & 0.2 \end{pmatrix}$, *i.e.* the columns 21 to 24 of K_3, $(K_3)_{2\ -2} = \begin{pmatrix} 5.6 & -2.4 & 2.2 & -0.4 \\ -0.3 & -0.3 & -0.2 & -0.1 \end{pmatrix}$, *i.e.* the columns 25 to 28 of K_3, $(K_3)_{2\ -3} = \begin{pmatrix} -4.4 & 1.8 & -3.2 & 0.4 \\ 0.1 & 0.1 & 0.2 & 0.0 \end{pmatrix}$, *i.e.* the columns 29 to 32 of K_3, $(K_3)_{3\ -3} = \begin{pmatrix} -19.8 & 6.6 & -3.7 & 1.2 \\ 1.9 & 0.9 & 0.2 & 0.4 \end{pmatrix}$, *i.e.* the columns 33 to 36 of K_3, $(K_3)_{3\ -4} =$

$\begin{pmatrix} 8.3 & -2.7 & 2.2 & -0.5 \\ -0.5 & -0.3 & -0.2 & -0.1 \end{pmatrix}$, *i.e.* the columns 37 to 40 of K_3, $(K_3)_{4\ -4} = \begin{pmatrix} -2.3 & 1.2 & -0.3 & 0.2 \\ -0.0 & 0.1 & 0.0 & 0.0 \end{pmatrix}$, *i.e.* the columns 41 to 44 of K_3, $(K_3)_{4\ -4} = \begin{pmatrix} 2.4 & -0.7 & 0.6 & -0.1 \\ -0.0 & -0.1 & -0.1 & -0.0 \end{pmatrix}$, *i.e.* the columns 45 to 48 of K_3, $(K_3)_{4\ -5} = \begin{pmatrix} -7.5 & 7.4 & 4.1 & 2.7 \\ -1.1 & 0.4 & 0.3 & 0.1 \end{pmatrix}$, *i.e.* the columns 49 to 52 of K_3, $(K_3)_{5\ -5} = \begin{pmatrix} -5.0 & 1.8 & -3.2 & 0.4 \\ 0.2 & 0.1 & 0.2 & 0.0 \end{pmatrix}$, *i.e.* the columns 53 to 56 of K_3, $(K_3)_{5\ -6} = \begin{pmatrix} 1.8 & -0.6 & 0.6 & -0.1 \\ 0.0 & -0.1 & -0.1 & -0.0 \end{pmatrix}$, *i.e.* the columns 57 to 60 of K_3, $(K_3)_{6\ -6} = \begin{pmatrix} -1.6 & 0.4 & -0.8 & 0.1 \\ 0.0 & 0.0 & 0.1 & 0.0 \end{pmatrix}$, *i.e.* the columns 61 to 64 of K_3, the elements of the first raw of the 2×256-rectangular matrix K_4 are 191.6; -624.3; -130.5; -121.3; -353.1; 269.2; 120.9; 64.6; -11.1; 8.8; -16.5; -0.1; -55.4; 56.8; 26.7; 13.5; -182.6; 223. 0; 120.5; 54.1; 56.8; -58.7; -84.7; -17.4; 21.3; -9.5; 11.5; -1. 0; 8.0; -12.2; -19.6; -3.8; -64.7; 21.3; -18.1; 3.2; 34.1; -12.6; 12.4; -1.9; -4.8; 2.7; -1.8; 0.4; 8.0; -3.0; 2.9; -0.5; -25.7; 45.7; 27.1; 10.9; 7.4; -11.2; -20.0; -3.4; 5.1; -2.5; 2.8; -0.3; 0.3; -2.1; -4.6; -0.7; -216.7; 222.6; 121.5; 54.1; 56.4; -55.5; -84.3; -16.7; 22.2; -9.1; 11.3; -0.9; 8.0; -11.5; -19.6; -3.6; -11.1; -32.9; -85.5; -11.2; 54.6; -13.8; 48.2; -1.3; -15.5; 6.0; -5.9; 0.9; 14.9; -4.0; 11.7; -0.4; 40.6; -15.0; 12.6; -2.6; -20.1; 7.5; -6.3; 1.4; 2.8; -1.1; 0.9; -0.2; -5.1; 1.9; -1.6; 0.4; -8.8; -5.4; -20.2; -2.1; 15.6; -4.4; 11.8; -0.5; -4.1; 1.6; -1.5; 0.3; 4.1; -1.2; 2.8; -0.2; -64.5; 28.0; -17.4; 4.7; 40.8; -15.7; 11.8; -2.6; -4.1; 2.1; -1.8; 0.3; 9.5; -3.8; 2.8; -0.6; 46.3; -18.3; 12.1; -3.3; -23.5; 8.5; -6.1; 1.6; 2.3; -0.9; 0.9; -0.1; -5.9; 2.2; -1.5; 0.4; -12.1; 3.5; -1.9; 0.6; 3.5; -0.9; 0.9; -0.1; -0.8; 0.3; -0.2; 0.1; 0.9; -0.3; 0.2; -0.0; 11.9; -4.6; 2.9; -0.8; -6.0; 2.2; -1.5; 0.4; 0.7; -0.3; 0.2; -0.0; -1.4; 0.5; -0.4; 0.1; -35.9; 45.4; 27.0; 10.8; 7.1; -10.3; -19.8; -3.2; 5.0; -2.2; 2.7; -0.2; 0.2; -1.9; -4.5; -0.6; -9.1; -5.2; -20.0; -2.0; 15.8; -4.3; 11.7; -0.5; -3.8; 1.5; -1.5; 0.2; 4.2; -1.2; 2.8; -0.2; 10.2; -3.7; 2.9; -0.6; -5.1; 1.9; -1.6; 0.3; 0.7; -0.3; 0.2; -0.0; -1.2; 0.4; -0.4; 0.1; -3.2; -0.6; -4.6; -0.3; 4.3; -1.3; 2.8; -0.2; -0.9; 0.4; -0.4; 0.1; 1.1; -0.4; 0.7; -0.1 respectively, and the elements of the second raw of the 2×256-rectangular matrix K_4 are -32.5; -42.7; -52.6; -12.2; -0.8; 15.6; 24.3; 4.8; 3.8; 1.3; 2.1; 0.4; 0.7; 3.1; 4.7; 0.9; 9.3; 12.9; 24.3; 4.2; 0.1; -2.5; -5.0; -0.8; -2.7; -0.8; -1.7; -0.3; -0.1; -0.5; -0.9; -0.1; 6.4; 1.1; 1.8; 0.5; -2.9; -1.3; -1.5; -0.4; 0.0; 0.3; 0.3; 0.1; -0.6; -0.3; -0.3; -0.1; 2.0; 2.5; 4.7; 0.8; -0.1; -0.4; -0.9; -0.1;-0.6; -0.2; -0.4; -0.1; -0.0; -0.1; -0.1; -0.0; 9.3; 13.7; 25.2; 4.4; 0.9; -2.4; -5.5; -0.7; -1.9; -1.3; -1.6; -0.4; 0.1; -0.4; -1.0; -0.1; 2.9; -1.8; -5.6; -0.6; -1.9; -1.3; -1.7; -0.5; 1.0; 0.7; 1.0; 0.2; -0.4; -0.4; -0.5; -0.1; -3.9; -1.1; -1.5; -0.4; 1.3; 0.9; 1.0; 0.3; 0.1; -0.2; -0.1; -0.1; 0.3; 0.2; 0.2; 0.1; 0.8; -0.3; -1.0; -0.1; -0.4; -0.4; -0.5; -0.1; 0.2; 0.2; 0.2; 0.1; -0.1; -0.1; -0.2; -0.0; 4.6; 1.7; 1.5; 0.6; -2.4; -1.5; -1.4; -0.5; -0.2; 0.4; 0.2; 0.1; -0.5; -0.4; -0.3; -0.1; -4.3; -1.0; -1.4; -0.4; 1.5; 0.8; 0.9; 0.3; 0.1; -0.2; -0.1; -0.1; 0.3; 0.2; 0.2; 0.1; 0.2; 0.3; 0.2; 0.1; 0.0; -0.2; -0.1; -0.1; -0.0; 0.0; 0.0; 0.0; 0.0; -0.0; -0.0; -0.0; -0.9; -0.2; -0.3; -0.1; 0.3; 0.2; 0.2; 0.1; 0.0; -0.0; -0.0; -0.0; 0.1; 0.0; 0.1; 0.0; 2.3;

2.7; 4.9; 0.8; 0.1; -0.4; -1.0; -0.1; -0.4; -0.3; -0.4; -0.1; 0.0; -0.0; -0.1; -0.0; 0.8; -0.3; -1.0; -0.1; -0.5; -0.4; -0.5; -0.1; 0.2; 0.2; 0.2; 0.1; -0.1; -0.1; -0.2; -0.0; -0.8; -0.3; -0.3; -0.1; 0.3; 0.2; 0.2; 0.1; 0.0; -0.0; -0.0; -0.0; 0.1; 0.1; 0.1; 0.0; 0.3; -0.0; -0.2; -0.0; -0.1; -0.1; -0.1; -0.0; 0.0; 0.0; 0.1; 0.0; -0.0; -0.0; -0.0; -0.0 respectively.

The matrices W_{ijk} are given by the equation (6.23) and V_{ij} given by (6.19). The definition of the matrices $P_{i(j)}$ is given by (6.12). The calculus of the matrix P_1 is given by the resolution of the ARE (6.26). The algorithm for the calculus of the matrix P_2 is given by the equations (6.27) to (6.49) and the algorithm of calculus of the matrices P_p, $p \geq 3$ is given by (6.50) to (6.81). The details of the algorithms are shown in Appendix E.

7.5 Simulation results

The results of the simulations are shown for the different orders of truncation $(i = 1, 2, 3, 4)$. In order to minimize the steady state errors, the performance index is minimized with the weighting matrices R and Q, where

$$R=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } Q=\begin{bmatrix} 200 & 0 & 0 & 0 \\ 0 & 200 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix} \quad (7.23)$$

Q and R have been used in [61] with the linearized optimal control.

The simulations have been applied for the linear control, Lin, where the dynamics is linearized about the origin and the proposed nonlinear controllers for the different orders of truncations (2^{nd}, 3^{rd} and 4^{th}). Note that the simulations were done for a desired yaw angle of 0 degree and a desired pitch angle of different values. For all simulations the initial condition of the pitch angle is -40.5 degree. The simulation results are summarized in Table 7.2. As a perspective for this work, we suggest to investigation of a guideline to select the best order of truncation of the optimal control. Now, the unique argument that justifies such a choice would be the computation limits (time and memory size). In fact, a second or third order control could be enough to improve the performance.

Table 7.2 Costs and steady state errors for different orders of truncations

Desired pitch angle in degree	**Cost by order**				**Steady state error by order**			
	1	**2**	**3**	**4**	**1**	**2**	**3**	**4**
Step 0 degree	066.8	068.1	115.2	081.5	00.0	00.0	00.0	00.0
Step -30 degree	275.0	287.5	301.9	298.1	04.5	04.4	03.5	03.7
Sin 0.05Hz -10 degree	115.2	114.9	204.9	139.7	01.1	01.1	01.1	01.1
Sqa 0.05Hz -10 degree	145.0	146.4	194.2	187.6	03.5	03.5	03.4	03.4
Sqa 0.02Hz -20 degree	188.8	190.9	255.3	256.7	12.2	12.0	10.6	10.4
Sin 0.02Hz -20 degree	130.8	131.4	156.9	137.4	03.9	03.8	02.9	03.2
Esc (multi steps)	105.7	105.8	107.4	107.6	04.5	04.4	03.8	03.8

Despite that there is no improvement of the cost using higher order controllers; we see that there are improvements in the steady state errors for all the desired trajectories except for the step 0 degree.

In the following we present the simulation results for a desired yaw angle of 0 degree, desired pitch angle of 0 degree and an initial condition of the pitch angle of -40.5 degrees for four controllers: Linear, 2^{nd} , 3^{rd} and 4^{th} truncation orders.

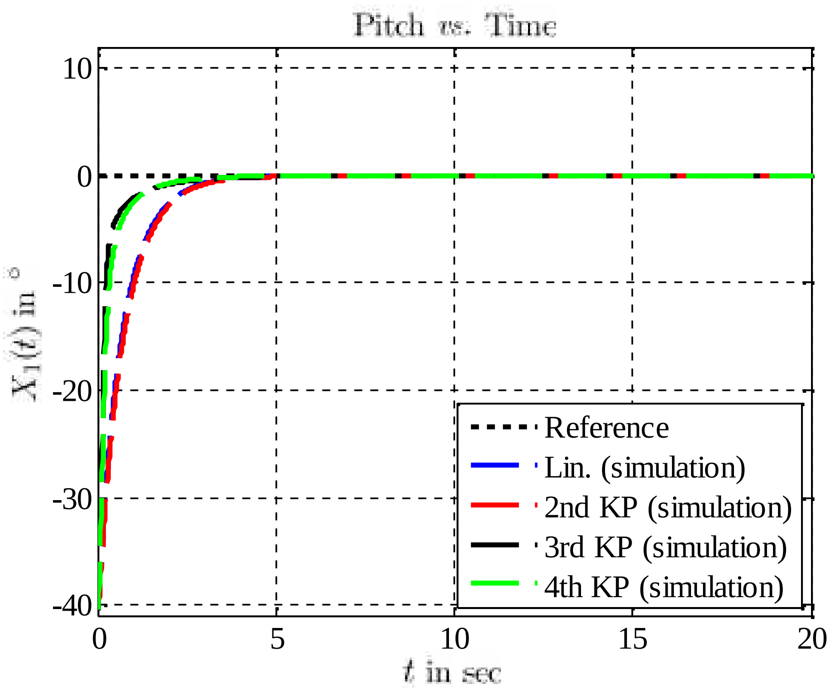

Figure 7.3 Pitch evolution vs. time for a desired pitch angle of 0 degree

The simulations show in Figure 7.3 that the four controllers stabilize the system at the desired pitch angle of 0 degree with an advantage for the 3^{rd} and 4^{th} ones in terms of rise time and settling time when compared to the Linear and 2^{nd} controllers. Those in Figure 7.4 show that the four controllers stabilize the system at the desired yaw angle of 0 degree with the advantage for the linear and 2^{nd} order ones in terms of settling time.

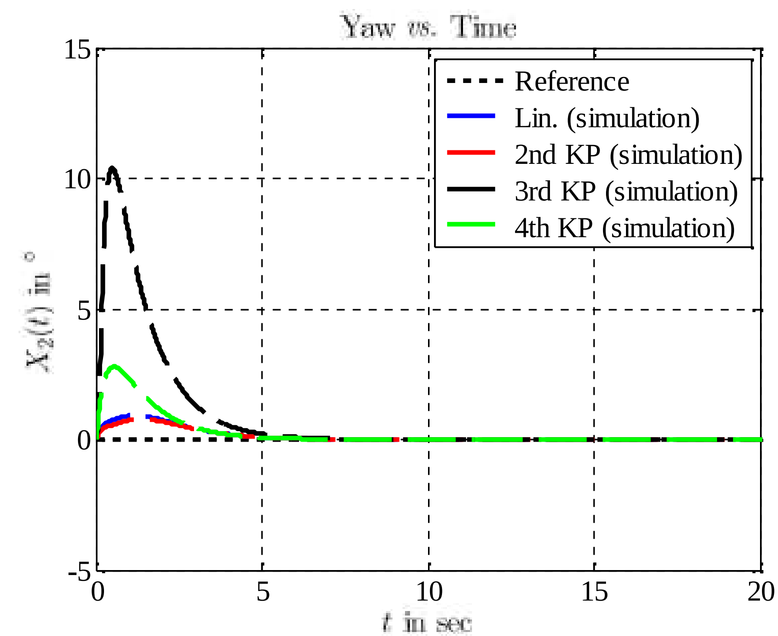

Figure 7.4 Yaw evolution *vs.* time for desired pitch angle of 0 degree

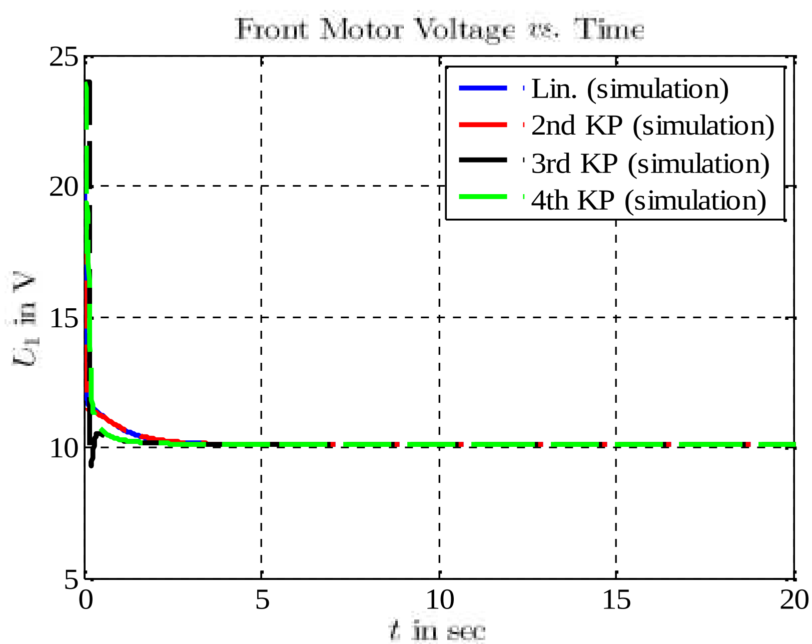

Figure 7.5 Front motor voltage evolution *vs.* time for desired pitch angle of 0 degree

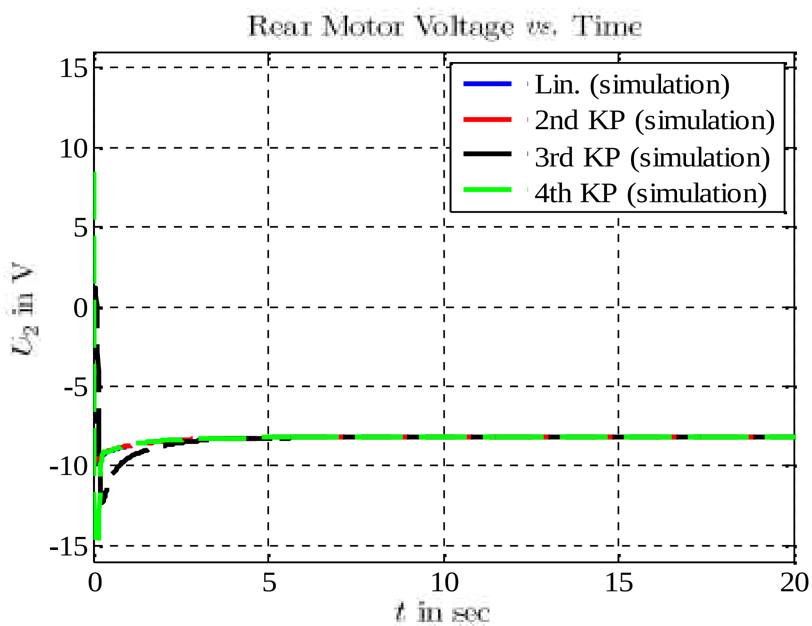

Figure 7.6 Rear motor voltage evolution *vs.* time for desired pitch angle of 0 degree

In terms of input control signals, Figures 7.5 and 7.6 show that the front motor voltage presents almost the same behaviour for the four controllers, and the rear motor voltage present the same behaviour with the first and second order controllers but the third and fourth ones present a higher voltage. In terms of cost, except for the third order controller which present a higher cost, the linear, second and fourth order controllers present a lower cost within same range.

More simulations have been tested and presented with different trajectories in Appendix F.

7.6 Experimental results

7.6.1 Experimental set-up presentation

The 2-DOF helicopter setup of Quanser Inc consists of four major components: the helicopter body, the power amplifiers, the data acquisition board and the real time control software. As mentioned in section 7.1 of this chapter and showed in Figure 7.1, the helicopter plant has six major components: two DC motors called the yaw and pitch motors which are actuating respectively the back and front propellers in which they have thrust force constants of $0.43\,N/V$ and $1.04\,N/V$, respectively. The two encoders are measuring respectively the yaw and pitch angles; the first one has 8192 counts per revolution and it has a position resolution of $0.0439\,\text{deg/count}$ and the second one has 4096 counts per revolution and a resolution of $0.791\,\text{deg/count}$ [61]. The two power amplifiers are two electronic modules in which they amplify and control the voltage of the pitch and yaw motors. These two amplifiers are called respectively UPM-2405 and UPM-1503, and shown in Figures 7.7 and 7.8.

The wiring system of Quanser 2-DOF set up is composed of six connection cables (numbered from1 to 6) that connect the plant to the PCB and from one cable (called J1) that connect the PCB to the computer. The function of each cable is summarized in Table 7.3 [61].

Figure 7.7 Pitch motor voltage amplifier (UPM-2405) [61]

Figure 7.8 Yaw motor voltage amplifier (UPM-1503) [61]

The data acquisition board consists of a Printed Circuit Board (PCB) and seven connection cables as shown in Figures 7.9 and 7.10.

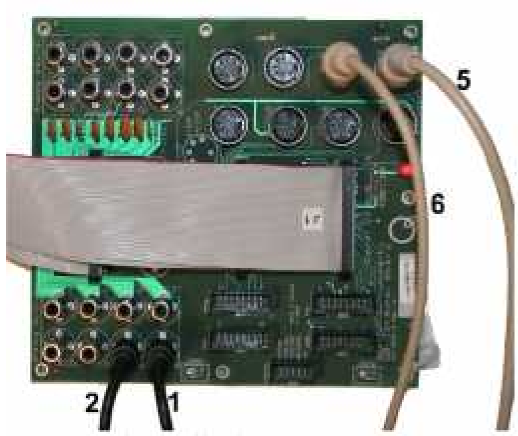

Figure 7.9 Printed Circuit Board of Quanser 2-DOF set up [61]

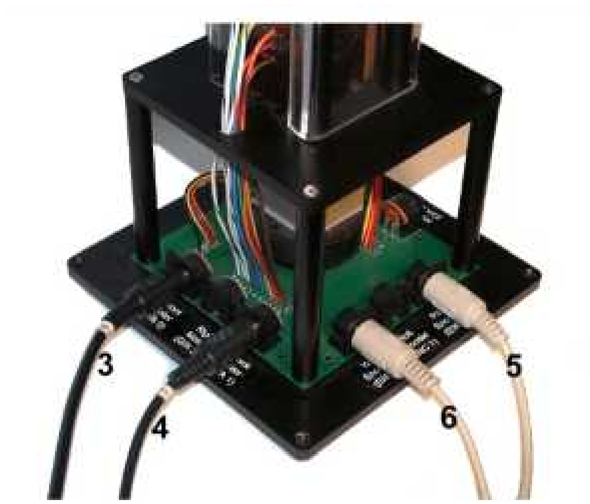

Figure 7.10 Wiring connections of Quanser 2-DOF set up [61]

Table 7.3 2-DOF Helicopter system wiring summary [61]

Cable#	From	To	Signal
1	Terminal Board: DAC#0	Front-UPM "From D/A" connector	Controls signal to the front UPM
2	Terminal Board: DAC#1	Back-UPM "From D/A" connector	Controls signal to the back UPM
3	UPM-2405 " to load connector"	2-DOF helicopter "Front motor D/A 0"	Power leads to the 2-DOF helicopter's front DC motor (propeller)
4	UPM-1503 " to load connector"	2-DOF helicopter "Back motor D/A 1"	Power leads to the 2-DOF helicopter's back DC motor (propeller)
5	2-DOF Helicopter " yaw encoder ENC 0"	Terminal Board:	2-DOF helicopter's yaw angle feedback signal to the data acquisition card
6	2-DOF Helicopter " pitch encoder ENC 1"	Terminal Board:	2-DOF helicopter's pitch angle feedback signal to the data acquisition card
7	Terminal Board	Computer	Transfer all data from terminal board to the computer

The real time software is a PC equipped with MATLAB®/SIMULINK® software, in which we can program the designed (Linear, 2^{nd} , 3^{rd} and 4^{th} order) controllers, then realize the experiments.

7.6.2 Experimental conditions

We consider the 2-DOF set-up of nonlinear dynamics given by (7.21) and the control law given by (7.22). The performance index is minimized using the weighting matrices given by (7.23). The experimental results are given for the linear controller, Lin, and for the nonlinear controllers of 2^{nd}, 3^{rd} and 4^{th} orders of truncations. Note that the experiments were done for a desired yaw angle of 0 degree and desired pitch angle of different trajectories, with an initial condition of the pitch angle of -40.5 degree.

7.6.3 Experimental results for desired pitch and yaw angles of 0 degree

In the following, we present the experimental results for a desired yaw angle of 0 degree, desired pitch angle of 0 degree and an initial condition of the pitch angle of -40.5 degrees for four controllers: Linear, 2^{nd}, 3^{rd} and 4^{th} truncation orders.

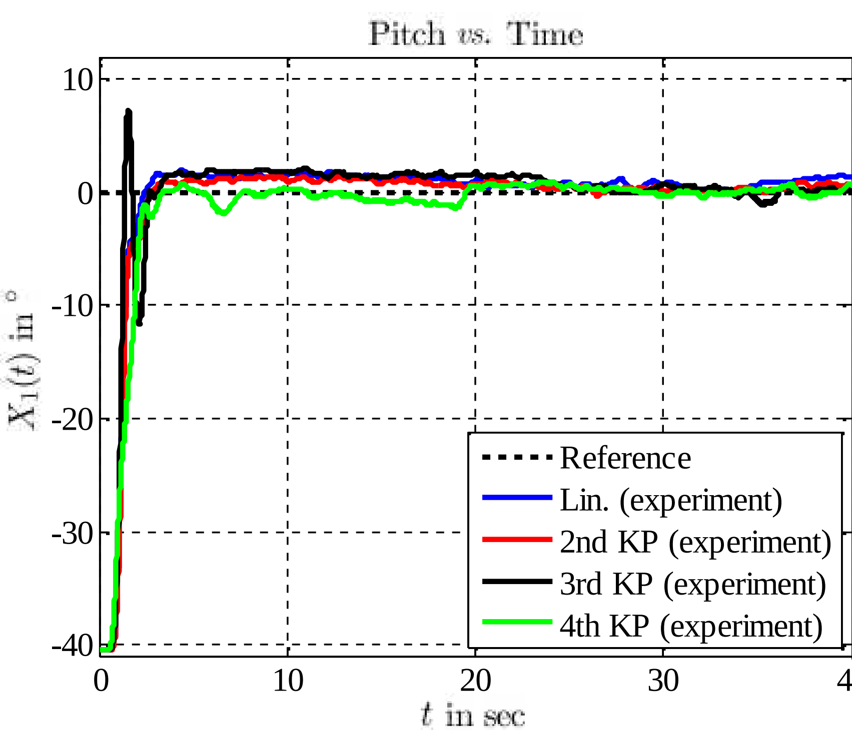

Figure 7.11 Pitch evolution *vs.* time for desired pitch angle of 0 degree

The experimental results show that the four controllers stabilize the 2-DOF helicopter set-up around the desired pitch angle of 0 degree despite of an important overshoot for the 3^{rd} order one. In comparison with the simulation results, the experimental ones present almost the same predicted behaviour for all the controllers except for the 3^{rd} order one which presents an important overshoot during the experiment.

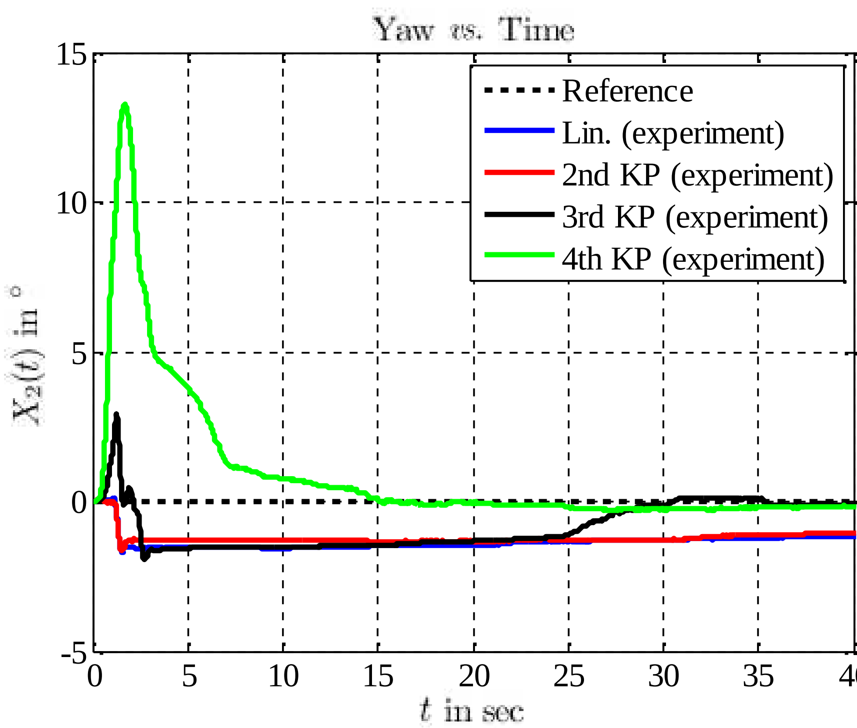

Figure 7.12 Yaw evolution *vs.* time for desired pitch angle of 0 degree

The experimental results show that the four controllers stabilize the 2-DOF helicopter set-up around the desired yaw angle of 0 degree despite the important overshoot for the 4^{th} order one. In comparison with the simulation results, the experimental ones present almost the same behaviour for the linear and 2^{nd} order controllers while the 3^{rd} order controller presents a lower overshoot and the 4^{th} order controller a higher overshoot than the simulation ones. We note the important discrepancy of the model representing the 2-DOF setup. In fact, the unmodeled dynamics, such as static and kinematic frictions, make the model representation on which depend our design less accurate. These uncertainties could be of great importance during the experiments and affect harmfully the performance of the proposed control.

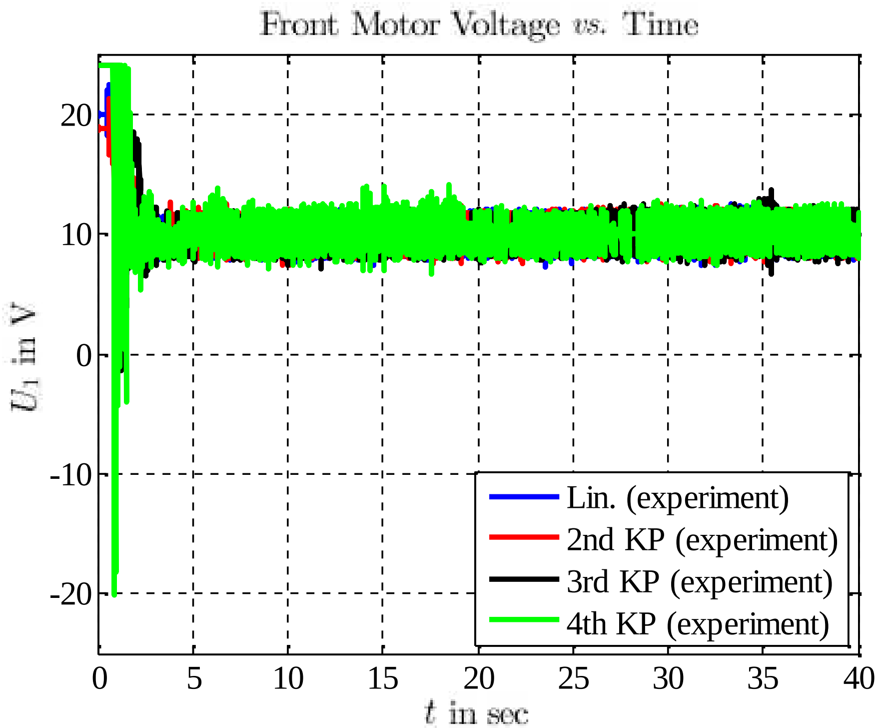

Figure 7.13 Front motor voltage *vs.* time for desired pitch angle of 0 degree

The experimental results for the input control, *i.e.*, front motor voltage show that from 0 to 2.3s, the four controllers require higher voltage and then energy. The 4^{th} order one has bigger fluctuations between -20 and 20V. After 2.3s, the four controllers behave almost the same way and have fluctuations in the range of 7 to 13V. In comparison with the simulation results, the experimental ones present higher fluctuations around the equilibrium voltage of 10V, this is due to the mathematical model of the control law which does not take into account the nonlinear terms higher than the 4^{th} order of truncations and some "noise" parameters. More experimental results have been tested and presented with different trajectories in Appendix G.

7.7 Conclusion

The main objective of this chapter was to test the effectiveness of the proposed new design through the run of simulations and experiments to a nonlinear dynamics: the 2-DOF helicopter model set-up of Quanser Inc. It is important to notice that the performance of the proposed control of higher order compete with the linear control one. All these controllers have been computed to run essentially the case where the equilibrium is at the origin for all states. All simulations and experiments conducted for different pitch targets and trajectories have been tested to evaluate the limits of the different controllers. In fact, accurate control gains would be recalculated for the different equilibriums to obtain consistent results.

8 Conclusion

Most of real plants are nonlinear and linear control strategy does not represent the best performance, have a limited domain of attraction and does not guarantee the stability of the closed loop systems. To overcome these issues, control engineers and researchers write the nonlinear dynamics in polynomial forms in terms of KP and then design an optimal nonlinear controller showing some interesting results. Since this controller does not guarantee the stability of the closed loop system, we had the idea to design a stabilizing one by choosing the cost minimum function to be computed in a quadratic form to satisfy the conditions of a Lyapunov candidate function. This new method was the main contribution of this work in addition to the theoretical framework related to KP algebra, optimal control theory and optimal control of polynomial systems, as well as an application to a real plant, the 2-DOF helicopter model.

The structure of this thesis was as follows, In chapter 1, we began by the introduction of this work, in which we presented the general context, its purpose and how it will be organized. In chapter 2, we presented the state of the art related to the main topics of this work. We began by a brief history of the optimal control theory, in which we showed the roots and the evolution of this theory from Newton 1685, when he presented a solution to the nose shape of a projectile providing minimum drug problem, to Bellman in 1950, who he established the necessary and sufficient conditions for the optimality through the popular HJB equation. Then, starting from the optimality condition, we cited more recent works, in which researchers presented numerous methods and algorithms to solve the optimal control problem for many classes of nonlinear systems. As any nonlinear function can be written in a polynomial form, the control design and the stability of polynomial systems was a subject treated too. We cited many recent works of design and analysis of the stability of nonlinear controllers. Due to the importance of the KP algebra in this work, we presented this framework and some of its applications in the last section of chapter 2. In chapter 3, we presented two major topics. The first one related to some basic definitions and proprieties of KP, $vec(\cdot)$ and $mat(\cdot)$ operators. What's new in this section is that we stated and proved two new theorems and three news lemmas useful in the following chapters. The second topic was related to the vector power series motivation, in which we recalled the multivariable Taylor expansion and applied this principle to three examples (scalar, two variables and second order dynamics) and plotted the exact function versus the different approximations to show the improvement and the best fitting obtained with high order approximations. Chapter 4 was dedicated to the optimal control theory. First, we showed its roots through the presentation of the optimization frame work, without and with equality constraints. In the optimization with no constraint section we presented the necessary and sufficient conditions of optimality and we illustrated with an example. Then, we stated the general optimization with no constraint problem, and we presented the necessary condition (which is the Euler-Lagrange equation in this case) to have a solution, next we illustrated with two examples. In the optimization with equality constraints section, after stating the problem, we cited three methods to solve the problem, two of them which are not detailed are the direct substitution method and the constraint variation method. The third method given in details is the Lagrange multipliers. For simplicity, we presented this method first in the case of two variable function to be minimized and one constraint, and we illustrated with an example. Then, we stated the problem formulation in case of integral functional and we finished by stating the problem of minimization of an integral functional subject to many constraints in its general form. Thereafter, we treated the general problem of optimal control theory and how the Lagrange multipliers lead to the canonical Hamilton equations. At this point two cases raised, problems with free final time (called infinite horizon) and problems with fixed final time (called finite horizon), the latter was illustrated with an example. The case of optimal control with equality constraints and for finite horizon was the subject of the following section, in which we treated the problem in details and we showed how this leads to the HJ equation problem. For the infinite horizon problems, we stated the case of linear time varying systems and the case of general time invariant systems in which we stated the optimal control laws and we illustrated with examples. In particular, we treated LTI systems in which we showed the design approach of one of the most popular controllers, the LQR in which the unknown part of its gain matrix is a direct solution of the ARE. Some useful definitions and proprieties related to the ARE were presented, the stability of the LQR was investigated and an illustrative example of the LQR was showed. In chapter 5, we treated the optimal control of a specific class of nonlinear systems written in a polynomial form in terms of KP. We stated the problem and we showed how it was transformed to solve a nonlinear equation. Since the latter is hard to solve analytically, some authors proposed an approximating method to write the unknown vector in a polynomial form using the KP tensor and by using its proprieties, the problem was transformed to calculate algebraic matrix equations. The next section was dedicated to the calculus of the unknown matrices, by cancelling the coefficients of the power series of the same exponent. We began with the first order which led to the calculus of the gain matrix of the linear controller by solving a classical ARE. Then, we presented the resolution process of the second order equation using some KP proprieties leading to the calculation algorithm of the second matrix gain in terms of the previous one and all other known matrices. Next, we presented the resolution process of the general order equation, which leads to solving an equation where two cases raised depending on the truncation order p even or odd. To illustrate the efficiency of this method, we applied the proposed design to three polynomial scalar examples, the F8 fighter model presented with a polynomial dynamics and to the Maglev set-up dynamics we approximated by a polynomial form of a third order using the Taylor series development. The application and the simulation of the KP-based controller to the three scalar examples showed an improvement of the proposed high order controllers (second and third order) versus the linear one, in terms of cost and interval of attraction. The application and simulation of the F8 fighter system showed an improvement in terms of interval of attraction. The third order KP controller stabilizes the system for an initial condition of 0.6 while the linear and the second order fail. The application and the simulation of the Maglev system showed an improvement in terms of domain of attraction too. The second order KP controller stabilizes the system for an initial condition equal to 0.050m, 0.075m and 0.100m while the linear and the third order KP controllers fail. Those simulations of the KP design to real plants (*i.e.*, F8 fighter and Maglev set-up) showed that the stability is not guaranteed with higher initial conditions. In fact, there is no theoretical framework that can show the stability of the discussed design approach. For this reason, we had the idea to design a stabilizing controller, by extending the previous work of KP design and choosing the cost function to be minimized in a quadratic form to satisfy the conditions of a Lyapunov function and to guarantee the asymptotic stability. This method, called KP-Lyapunov-Function-based design or simply KP-LF one, was the subject of the chapter 6. After the introduction of this chapter, we stated the optimal control problem and we showed how this problem was transformed into solving a nonlinear differential equation. Different from the KP method in this one, we approximated the cost function by a quadratic form and we re-wrote the differential equation to be solved. Then, through the application of appropriate algebraic operators and using new KP proprieties introduced in chapter 3, we wrote the same equation in a more compact form in unknown matrices and a real scalar. The resolution of this equation to find these matrices by cancelling the coefficients of the power state vector was introduced. The procedure resembles the previous one to compute these unknown terms leading to an ARE and first order algebraic equation problems. We used also some KP proprieties and introduced the non-redundant vector power to overcome singularity issues. Thereafter, we discussed the stability of the closed loop system controlled by the LF design, and we showed that the closed loop system could be ideally globally asymptotically stable. To illustrate the efficiency of the LF method we presented the application of two examples. For the scalar example, the simulation results showed the cost improvement obtained with LF design (for both second and third order) versus the KP and linear controllers. The simulation of the F8 fighter showed an improvement obtained by the LF controllers versus KP and linear controllers. This improvement was obtained in terms of cost reduction as well as a larger domain of attraction (for different initial conditions, the LF controller stabilizes the system, while the KP and linear controllers fail). As the real world may be slightly different from the simulation and theoretical world, we treated in chapter 7 the application of the proposed LF method to an experimental set-up of a 2-DOF helicopter model of Quanser Inc. After the introduction, we presented a brief description of the system and cited its mechanical and electrical proprieties. Then, we introduced the dynamics of the system. The free body diagram of the set-up which allowed writing its equation of motion by expressing the kinetic and potential energies, and the application of the Lagrangian were presented in Appendix A. Then, we wrote the state space dynamics of the system, calculated the equilibrium and presented its linearized and approximated polynomials (of second, third and fourth orders) using the Taylor vector power expansion. Thereafter, we presented the control design and referred to Appendix B for more details regarding the algorithm of calculus of the gain matrices. The simulations of the proposed controllers, for different orders of truncation (linear, second, third and fourth) and for different desired trajectories, were presented. The simulation results showed some interesting results for some nonlinear controllers. The realization of the experiments for the same desired trajectories showed almost the same predicted behaviour obtained through the simulations with the exception for some controllers which they presented a slightly different behaviour (higher overshoot), which is due (in our point of view) to the errors occurred by the approximation of the equation of motion and some uncontrollable noises.

The experimental results remain biased due to discrepancies of the nonlinear model representing the setup. We believe that the modelling errors affect the control performances in particular with the higher orders. More accurate model estimation would be developed to match the model with the setup dynamics. Finally we would note that despite the theoretical framework of stability discussed in [57], there is no study to estimate the domain of attraction obtained by the LF method. This topic can be a subject to future research.

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Appendices

A Proofs of theorems and lemmas

A.1 Proof of Theorem 3.18

We have

$$U_{n^p\times n} = U_{\left(n\cdot n^{p-1}\right)\times n} \tag{A1.1}$$

$$= U_{n\times n^p}\cdot U_{n^{p-1}\times n^2} \tag{A1.2}$$

Multiplying the equation (A1.1) by $U_{n^p\times n}$ and using Theorem 3.1, we obtain

$$U^2_{n^p\times n} = U_{n^{p-1}\times n^2} = U_{\left(n\times n^{p-2}\right)\times n^2} \tag{A1.3}$$

$$= U_{n\times n^p}\cdot U_{n^{p-2}\times n^3} \tag{A1.4}$$

Multiplying the equation (A1.3) by $U_{n^p\times n}$ and using Theorem 3.1, we obtain

$$U^3_{n^p\times n} = U_{n^{p-2}\times n^3} \tag{A1.5}$$

By repeating the same procedure, we obtain

$$U^{p-1}_{n^p\times n} = U_{n^2\times n^{p-1}} = U_{n\cdot n\times n^{p-1}} \tag{A1.6}$$

$$= U_{n\times n^p}\cdot U_{n\times n^p} \tag{A1.7}$$

Thus

$$U^p_{n^p\times n} = U_{n\times n^p} \tag{A1.8}$$

By applying Theorem 3.16 to the equation (A1.8), we can write, for $p = 2q$

$$U_{n^p\times n} = U_{n^{2q}\times n} = U_{n^q n^q\times n} = U_{n^q\times n^{q+1}}\cdot U_{n^q\times n^{q+1}} = \left(U_{n^q\times n^{q+1}}\right)^2 = \left(U_{n\cdot n^{q-1}\times n^{q+1}}\right)^2 \tag{A1.9}$$

$$= \left(U_{n \cdot n^{2q}} \cdot U_{n^{q-1} \times n^{q+1}}\right)^2 = \left(U_{n \times n^{2q}} \cdot U_{n \cdot n^{q-2} \times n^{q+2}}\right)^2 = U^2_{n \times n^{2q}} \cdot \left(U_{n.n^{q-2} \times n^{q+2}}\right)^2 \quad \text{(A1.10)}$$

$$= U^2_{n \times n^{2q}} \cdot \left(U_{n \times n^{2q}} \cdot U_{n^{q-2} \times n^{q+3}}\right)^2 = U^4_{n \times n^{2q}} \cdot \left(U_{n^{q-2} \times n^{q+3}}\right)^2 = U^4_{n \times n^{2q}} \cdot \left(U_{n \cdot n^{q-3} \times n^{q+3}}\right)^2 \quad \text{(A1.11)}$$

$$= U^4_{n \times n^{2q}} \cdot \left(U_{n^{q-2} \times n^{q+3}}\right)^2 = U^4_{n \times n^{2q}} \cdot \left(U_{n \cdot n^{q-3} \times n^{q+3}}\right)^2 = U^4_{n \times n^{2q}} \cdot \left(U_{n \times n^{2q}} \cdot U_{n^{q-3} \times n^{q+4}}\right)^2 \quad \text{(A1.12)}$$

$$= U^6_{n \times n^{2q}} \cdot \left(U_{n^{q-3} \times n^{q+4}}\right)^2 = \cdots = U^{2(q-1)}_{n \times n^{2q}} \cdot \left(U_{n^{q-(q-1)} \times n^{2q}}\right)^2 = U^{2(q-1)}_{n \times n^{2q}} \cdot U^2_{n \times n^{2q}} = U^{2q}_{n \times n^{2q}} \quad \text{(A1.13)}$$

Thus, we obtain

$$U_{n^p \times n} = U^p_{n \times n^p} \quad \text{(A1.14)}$$

A.2 Proof of Theorem 3.19

Two cases arise.

- case p even:

Assume (-1) is an eigenvalue of $U_{n^p \times n}$. Then it exists $v \neq 0$ such that

$$U_{n^p \times n} \cdot v = -v \quad \text{(A2.1)}$$

Using Theorem 3.1, by pre-multiplying (A2.1) by $U_{n \times n^p}$, that is,

$$U_{n \times n^p} \cdot U_{n^p \times n} \cdot v = -U_{n \times n^p} \cdot v \quad \text{(A2.2)}$$

That is,

$$U_{n \times n^p} \cdot v = -v \quad \text{(A2.3)}$$

Using Theorem 3.17, we have

$$U_{n^p \times n} \cdot v = U^{p}_{n \times n^p} \cdot v = U^{p-1}_{n \times n^p} \cdot U_{n \times n^p} \cdot v = -U^{p-1}_{n \times n^p} \cdot v \tag{A2.4}$$

$$= -U^{p-2}_{n \times n^p} \cdot U_{n \times n^p} \cdot v = \left(-1\right)^2 \cdot U^{p-2}_{n \times n^p} \cdot v \tag{A2.5}$$

$$= \cdots = \left(-1\right)^{p-1} \cdot U_{n \times n^p} \cdot v = \left(-1\right)^{p} \cdot v = \left(-1\right)^{2q} \cdot v \tag{A2.6}$$

That is,

$$U_{n^p \times n} \cdot v = v \tag{A2.7}$$

From (A2.1) and (A2.3), we have

$$v = -v \tag{A2.8}$$

That means $v = 0$, which is impossible as v is selected nonzero. So $\left(-1\right)$ is not an eigenvalue of $U_{n^p \times n}$, that is,

$$\left| U_{n^p \times n} + I_{n^{p+1}} \right| \neq 0 \tag{A2.9}$$

Thus, $\left(U_{n^p \times n} + I_{n^{p+1}} \right)$ is regular.

- case p odd:

Let a be an eigenvalue of $U_{n^p \times n}$ and $v \neq 0$ the corresponding eigenvector, that is,

$$U_{n^p \times n} \cdot v = a \cdot v \tag{A2.10}$$

Using Theorem 3.17, we write

$$U^{p}_{n \times n^p} \cdot v = a \cdot v \tag{A2.11}$$

Multiplying the equation (A2.12) by $U^{p}_{n^p \times n}$, we obtain

$$a \cdot U^{p}_{n^p \times n} \cdot v = v \tag{A2.12}$$

Using (A2.10), we have

$$a \,.\, a^{p} \cdot v = a^{p+1} \cdot v = v \quad \text{(A2.13)}$$

As $v \neq 0$, (A2.13) leads to

$$a^{p+1} = 1 \quad \text{(A2.14)}$$

p is odd, then $a \in \{1, -1\}$.

Identically, we obtain $a \in \{1, -1\}$ is an eigenvalue of $U_{n \times n^p}$. Note from Theorem 3.17

$$U_{n^p \times n} = U^{p}_{n \times n^p} \quad \text{(A2.15)}$$

Thus,

$$U^{p+1}_{n \times n^p} = I_{n^{p+1}} \quad \text{(A2.16)}$$

Consider $v \neq 0$ an eigenvalue of $U_{n \times n^p}$ associated with $a \in \{1, -1\}$. For $\sim$ eigenvalue of $U_{n^p \times n}$, we have

$$U_{n^p \times n} \cdot v + \sim \cdot v = U_{n^p \times n} \cdot v + \sim \cdot U^{p+1}_{n \times n^p} \cdot v = U_{n^p \times n} \cdot v + \sim \cdot a^{p+1} \cdot v = 0 \quad \text{(A2.17)}$$

Using (A2.14), we obtain

$$U_{n^p \times n} \cdot v + \sim \cdot v = U^{p}_{n \times n^p} \cdot v + \sim \cdot a^{p+1} \cdot v = a^{p} \cdot v + \sim \cdot a^{p+1} \cdot v = a^{p}\left(1 + \sim a\right) \cdot v = 0 \quad \text{(A2.18)}$$

Note that $a \in \{1, -1\}$ and $v \neq 0$, then $1 + \sim a = 0$. Assuming $a = 1$ leads to $\sim = -1$ eigenvalue of $U_{n^p \times n}$ and if $a = -1$, then (-1) is eigenvalue of $U_{n^p \times n}$.

A.3 Proof of Lemma 3.1

We have

- For $j = 1$, using Theorem 3.15, we have

$$\frac{\partial x^{|1|}}{\partial x^T} = \frac{\partial x}{\partial x^T} = I_n = x^{|0|} \otimes I_n \otimes x^{|0|} \quad \text{(A3.1)}$$

As $x^{|0|} = 1$. Since $D_1^{(n)} = I_n$, then

$$\frac{\partial x}{\partial x^T} = I_n = D_1^{(n)} = D_1^{(n)} \cdot \left(I_n \otimes x^{|0|} \right) \quad \text{(A3.2)}$$

- For $j = 2$, we write

$$\frac{\partial x^{|2|}}{\partial x^T} = \frac{\partial}{\partial x^T}\left(x \otimes x \right) \quad \text{(A3.3)}$$

Using Theorem 3.13, we can write

$$\frac{\partial}{\partial x^T}\left(x \otimes x \right) = \frac{\partial x}{\partial x^T} \otimes x + \left(I_1 \otimes U_{n\times n} \right)\left(\frac{\partial x}{\partial x^T} \otimes x \right)\left(I_n \otimes U_{1\times 1} \right) \quad \text{(A3.4)}$$

$$= I_n \otimes x + U_{n\times n}\left(I_n \otimes x \right) I_n = \left(I_{n^2} + U_{n\times n} \right)\left(I_n \otimes x \right) \quad \text{(A3.5)}$$

$$= \left(U_{n^0\times n} \otimes I_n + U_{n\times n} \otimes I_{n^0} \right)\left(I_n \otimes x \right) \quad \text{(A3.6)}$$

$$= \left(\sum_{i=0}^{1} U_{n^i\times n} \otimes I_{n^{1-i}} \right)\left(I_n \otimes x \right) \quad \text{(A3.7)}$$

$$= D_2^{(n)}\left(I_n \otimes x^{|2-1|} \right) \quad \text{(A3.8)}$$

- For $j = 3$, we write

$$\frac{\partial x^{|3|}}{\partial x^T} = \frac{\partial}{\partial x^T}\left(x^{|2|} \otimes x \right) \quad \text{(A3.9)}$$

Using Theorem 3.13, we can write

$$\frac{\partial}{\partial x^T}\left(x^{|2|}\otimes x\right)=\frac{\partial x^{|2|}}{\partial x^T}\otimes x+\left(I_1\otimes U_{n^2\times n}\right)\left(\frac{\partial x}{\partial x^T}\otimes x^{|2|}\right)\left(I_n\otimes U_{1\times 1}\right) \tag{A3.10}$$

$$=\frac{\partial x^{|2|}}{\partial x^T}\otimes x+U_{n^2\times n}\left(\frac{\partial x}{\partial x^T}\otimes x^{|2|}\right)U_{n\times 1}=\frac{\partial x^{|2|}}{\partial x^T}+U_{n^2\times n}\left(I_n\otimes x^{|2|}\right) \tag{A3.11}$$

$$=\left[\left(I_{n^2}+U_{n\times n}\right)\left(I_n\otimes x\right)\right]\otimes x+U_{n^2\times n}\left(I_n\otimes x^{|2|}\right) \tag{A3.12}$$

$$=\left[\left(I_{n^2}+U_{n\times n}\right)\otimes I_n\right]\left(I_n\otimes x\otimes x\right)+U_{n^2\times n}\left(I_n\otimes x^{|2|}\right) \tag{A3.13}$$

$$=\left[I_{n^3}+U_{n\times n}\otimes I_n+U_{n^2\times n}\right]\left(I_n\otimes x^{|2|}\right) \tag{A3.14}$$

$$=\left(U_{n^0\times n}\otimes I_{n^2}+U_{n\times n}\otimes I_n+U_{n^2\times n}\otimes I_{n^0}\right)\left(I_n\otimes x^{|2|}\right) \tag{A3.15}$$

$$=\left(\sum_{i=0}^{3}U_{n^i\times n}\otimes I_{n^{3-i-1}}\right)\cdot\left(I_n\otimes x^{|3-1|}\right)=D_3^{(n)}\left(I_n\otimes x^{|3-1|}\right) \tag{A3.16}$$

- Thus, for any nonzero integer j, we write

$$\frac{\partial x^{|j|}}{\partial x^T}=\frac{\partial}{\partial x^T}\left(x^{|j-1|}\otimes x\right) \tag{A3.17}$$

$$=\frac{\partial x^{|j-1|}}{\partial x^T}\otimes x+\left(I_1\otimes U_{n^{j-1}\times n}\right)\left(\frac{\partial x}{\partial x^T}\otimes x^{|j-1|}\right)I_n \tag{A3.18}$$

$$=\frac{\partial x^{|j-1|}}{\partial x^T}\otimes x+U_{n^{j-1}\times n}\left(I_n\otimes x^{|j-1|}\right)U_{n\times 1} \tag{A3.19}$$

$$=\frac{\partial x^{|j-1|}}{\partial x^T}\otimes x+x^{|j-1|}\otimes I_n\otimes x^{|0|} \tag{A3.20}$$

$$=\left(\frac{\partial x^{|j-2|}}{\partial x^T}\otimes x+x^{|j-2|}\otimes I_n\otimes x^{|0|}\right)\otimes x+x^{|j-1|}\otimes I_n\otimes x^{|0|} \tag{A3.21}$$

$$= \frac{\partial x^{|j-2|}}{\partial x^T} \otimes x^{|2|} + x^{|j-2|} \otimes I_n \otimes x^{|1|} + x^{|j-1|} \otimes I_n \otimes x^{|0|} \quad \text{(A3.22)}$$

$$= \left(\frac{\partial x^{|j-3|}}{\partial x^T} \otimes x + x^{|j-3|} \otimes I_n \otimes x^{|0|} \right) \otimes x^{|2|} + x^{|j-2|} \otimes I_n \otimes x^{|1|} + x^{|j-1|} \otimes I_n \otimes x^{|0|} \quad \text{(A3.23)}$$

$$= \frac{\partial x^{|j-3|}}{\partial x^T} \otimes x^{|3|} + x^{|j-3|} \otimes I_n \otimes x^{|2|} + x^{|j-2|} \otimes I_n \otimes x^{|1|} + x^{|j-1|} \otimes I_n \otimes x^{|0|} \quad \text{(A3.24)}$$

$$= x^{|0|} \otimes I_n \otimes x^{|j-1|} + x^{|1|} \otimes I_n \otimes x^{|j-2|} + x^{|2|} \otimes I_n \otimes x^{|j-3|} + \ldots + x^{|j-1|} \otimes I_n \otimes x^{|0|} \quad \text{(A3.25)}$$

$$= \sum_{i=0}^{j-1} x^{|i|} \otimes I_n \otimes x^{|j-i-1|} \quad \text{(A3.26)}$$

$$= \sum_{i=0}^{j-1} \left[U_{n^i \times n} \left(I_n \otimes x^{|i|} \right) I_n \right] \otimes x^{|j-i-1|} \quad \text{(A3.27)}$$

$$= \sum_{i=0}^{j-1} \left[U_{n^i \times n} \left(I_n \otimes x^{|i|} \right) \otimes \left(I_{n^{j-i-1}} x^{|j-i-1|} \right) \right] \quad \text{(A3.28)}$$

Using Theorem 3.5, we obtain

$$\frac{\partial x^{|j|}}{\partial x^T} = \sum_{i=0}^{j-1} \left[\left(U_{n^i \times n} \otimes I_{n^{j-i-1}} \right) \left(I_n \otimes x^{|i|} \otimes x^{|j-i-1|} \right) \right] \quad \text{(A3.29)}$$

$$= \left(\sum_{i=0}^{j-1} U_{n^i \times n} \otimes I_{n^{j-i-1}} \right) \left(I_n \otimes x^{|j-1|} \right) \quad \text{(A3.30)}$$

$$= D_j^{(n)} \left(I_n \otimes x^{|j-1|} \right) \quad \text{(A3.31)}$$

A.4 Proof of Lemma 3.2

Let $A=\begin{pmatrix} A_1 \\ \vdots \\ \vdots \\ \vdots \\ A_n \end{pmatrix}$ be partition of n blocks with $A_i \in \mathbb{R}^k \times \mathbb{R}^l$. We write

$$A_i = \begin{pmatrix} 0_k & \cdots & 0_k & I_k & 0_k & \cdots & 0_k \end{pmatrix} \begin{pmatrix} A_1 \\ \vdots \\ A_{i-1} \\ A_i \\ A_{i+1} \\ \vdots \\ A_n \end{pmatrix} \tag{A4.1}$$

where 0_k is the null square matrix of order k. I_k is the identity of order matrix in the i^{th} block. We note

$$\begin{pmatrix} 0_k \\ \vdots \\ 0_k \\ I_k \\ 0_k \\ \vdots \\ 0_k \end{pmatrix} = e_i^{(n)} \otimes I_k \tag{A4.2}$$

where $e_i^{(k)}$ is the n-dimensional unit column vector which is "1" in the i^{th} element and zero elsewhere, introduced in Definition 3.2. We write $\forall i = 1, \cdots, n$

$$vec\left(A_i^T\right) = vec\left[A^T\left(e_i^{(n)} \otimes I_k\right)\right] \tag{A4.3}$$

Using the second equality of Theorem 3.10, we obtain

$$vec\left(A_i^T\right)=\left[\left(e_i^{(n)T}\otimes I_k\right)\otimes I_l\right]vec\left(A^T\right)=\left(e_i^{(n)T}\otimes I_{kl}\right)vec\left(A^T\right)\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix} \tag{A4.4}$$

Now, we write

$$\left(I_n\otimes x^T\right)Ay=\begin{pmatrix}x^T & & 0\\ & \ddots & \\ 0 & & x^T\end{pmatrix}\begin{pmatrix}A_1\\ \vdots\\ A_n\end{pmatrix}y \tag{A4.5}$$

$$=\begin{pmatrix}x^TA_1y\\ \vdots\\ x^TA_ny\end{pmatrix}=\begin{pmatrix}y^TA_1x\\ \vdots\\ y^TA_nx\end{pmatrix}=\begin{pmatrix}vec^T\left(y^TA_1x\right)\\ \vdots\\ vec^T\left(y^TA_nx\right)\end{pmatrix} \tag{A4.6}$$

Using Theorem 3.7, the equation (A4.6) can be given by

$$\left(I_n\otimes x^T\right)Ay=\begin{pmatrix}vec^T\left(A_1^T\right)\left(x\otimes y\right)\\ \vdots\\ vec^T\left(A_n^T\right)\left(x\otimes y\right)\end{pmatrix}=\begin{pmatrix}vec^T\left(A_1^T\right)\\ \vdots\\ vec^T\left(A_n^T\right)\end{pmatrix}\left(x\otimes y\right) \tag{A4.7}$$

We substitute (A4.7) into (A4.5) to obtain

$$\left(I_n\otimes x^T\right)Ay=\begin{pmatrix}vec^T\left(A^T\right)\left(e_1^{(n)}\otimes I_{kl}\right)\\ \vdots\\ vec^T\left(A^T\right)\left(e_n^{(n)}\otimes I_{kl}\right)\end{pmatrix}=\left[I_n\otimes vec^T\left(A^T\right)\right]\left[\begin{pmatrix}e_1^{(n)}\\ \vdots\\ e_n^{(n)}\end{pmatrix}\otimes I_{kl}\right]\left(x\otimes y\right) \tag{A4.8}$$

Noting that

$$\begin{pmatrix}e_1^{(n)}\\ \vdots\\ e_n^{(n)}\end{pmatrix}=vec\left(I_n\right) \tag{A4.9}$$

we write

$$\left(I_n \otimes x^T\right)Ay = \left(I_n \otimes vec^T(A)\right)\left(vec(I_n)\otimes I_{kl}\right)(x\otimes y) \tag{A4.10}$$

A.5 Proof of Lemma 3.3

We have

$$\left(I_n \otimes vec^T(A)\right)\left(vec(I_n)\otimes I_{pq}\right) = \left(I_n \otimes vec^T\left[A_1 \quad \cdots \quad A_n\right]\right)\left(vec(I_n)\otimes I_{pq}\right) \tag{A5.1}$$

$$= \left(I_n \otimes \left[vec^T(A_1)\cdots \quad vec^T(A_n)\right]\right)\left(vec(I_n)\otimes I_{pq}\right) \tag{A5.2}$$

$$= \begin{pmatrix} \left[vec^T(A_1)\cdots \; vec^T(A_n)\right] & & 0 \\ & \ddots & \\ 0 & & \left[vec^T(A_1)\cdots \; vec^T(A_n)\right] \end{pmatrix} \begin{pmatrix} I_{pq} \\ \vdots \\ \vdots \\ \vdots \\ I_{pq} \end{pmatrix} \tag{A5.3}$$

$$= \begin{pmatrix} vec^T(A_1) \\ \vdots \\ vec^T(A_n) \end{pmatrix} = \left[vec(A_1)\cdots vec(A_n)\right]^T \tag{A5.4}$$

$\forall i = 1,\cdots,n \quad vec(A_i)$ is a column-vector of dimension pq, then

$$\left[vec(A_1)\cdots vec(A_n)\right] = mat_{pq\times n}\begin{pmatrix} vec(A_1) \\ \vdots \\ vec(A_n) \end{pmatrix} = mat_{pq\times n}\left[vec(A_1\cdots A_n)\right] = mat_{pq\times n}\left[vec(A)\right] \tag{A5.5}$$

Thus,

$$\left(I_n \otimes vec^T(A)\right)\left(vec(I_n)\otimes I_{pq}\right) = mat^T_{pq\times n}\left(vec(A)\right) \tag{A5.6}$$

B Illustrative examples of VPS and Taylor expansion

B.1 Example 3.1

We consider the scalar function $f(x) = e^x$. We denote by f_i, $i = 1, 3, 5$ and 7, the Taylor development functions of the order i. Using (A5.1), the calculations of the different approximation functions are

$$f_1 = 1 + x \tag{B1.1}$$

$$f_2 = 1 + x + \frac{1}{2}x^2 \tag{B1.2}$$

$$f_3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \tag{B1.3}$$

$$f_4 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 \tag{B1.4}$$

$$f_5 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 \tag{B1.5}$$

$$f_6 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 \tag{B1.6}$$

$$f_7 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 \tag{B1.7}$$

The graphic representations of the function f and the polynomial approximation f_1, f_3, f_5 and f_7 in the interval $[-2, 2]$ are shown in Figure B.1.

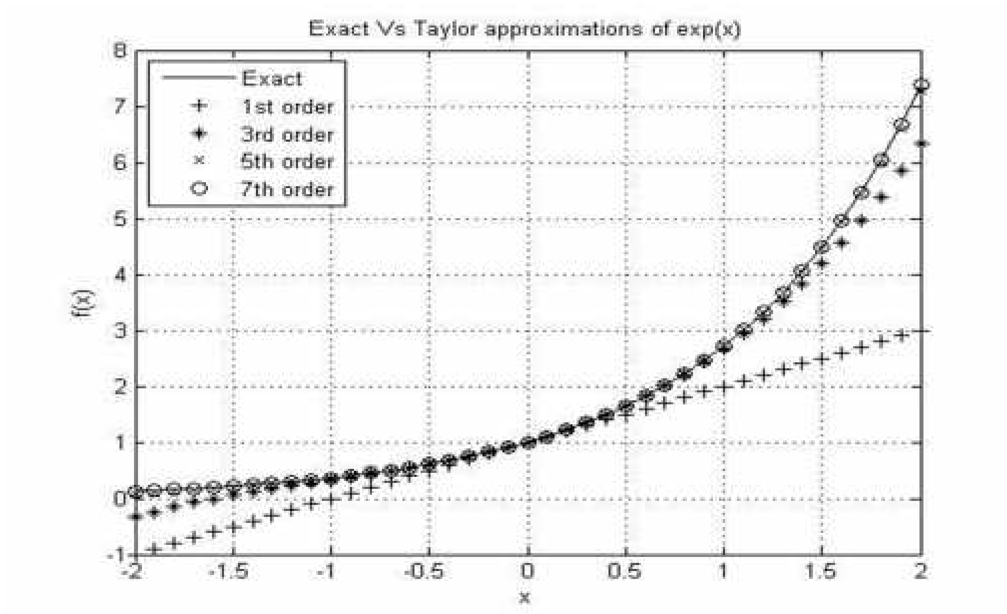

Figure B.1: Exact *vs.* Taylor approximations of e^x

We note in Figure B.1 that as much as the order of truncation goes up, the fitting of the exact function curve and the approximation function curve is better, and the interval of attraction is larger. With the different approximations (B1.1) to (B1.7), the magnitude of the interval of a best fitting increases with the order of truncation. In fact, for a best approximation with accuracy of less than 2% of the exact value of $f(x)$, we simulate numerically the different ranges shown in Table B.1.

Table B.1 Order of truncation *vs.* interval of best fitting of e^x within $\pm 2\%$ of accuracy

Order of Truncation	Range of fitting curves about $x = 0$	
	$x_{\min}$	$x_{\max}$
1	-0.19	0.21
2	-0.44	0.51
3	-0.72	1.02
4	-1.01	1.53
5	-1.29	2.09
6	-1.58	2.68
7	-1.87	3.31

B.2 Example 3.2

Consider the two-variable real valued function

$$f\left(x_1, x_2\right) = a\tan\left(x_1^2 + x_2^2\right) \cdot \cos\left(x_1^2 + \frac{\circledR}{6}\right) \tag{B2.1}$$

Using (A5.1), the calculus of the different approximation functions of order 2, 4 and 6 are

$$f_2\left(x_1, x_2\right) = \frac{\sqrt{3}}{2}\left(x_1^2 + x_2^2\right) \tag{B2.2}$$

$$f_4\left(x_1, x_2\right) = \frac{\sqrt{3}}{2}\left(x_1^2 + x_2^2\right) - \frac{1}{2}x_1^4 - \frac{1}{2}\left(x_1^2 \cdot x_2^2\right) \tag{B2.3}$$

$$f_6\left(x_1, x_2\right) = \frac{\sqrt{3}}{2}\left(x_1^2 + x_2^2\right) - \frac{1}{2}\left(x_1^4 + x_1^2 \cdot x_2^2\right) - \frac{5}{4\sqrt{3}}x_1^6 - \frac{3\sqrt{3}}{4}x_1^2 x_2^4 - \frac{1}{2\sqrt{3}}x_2^6 \tag{B2.4}$$

The graphic representation of the function f and its polynomial approximation f_2, f_4 *and* f_6 in the domain $[-0.5, 0.5] \times [-0.5, 0.5]$ are shown in Figure B.2.

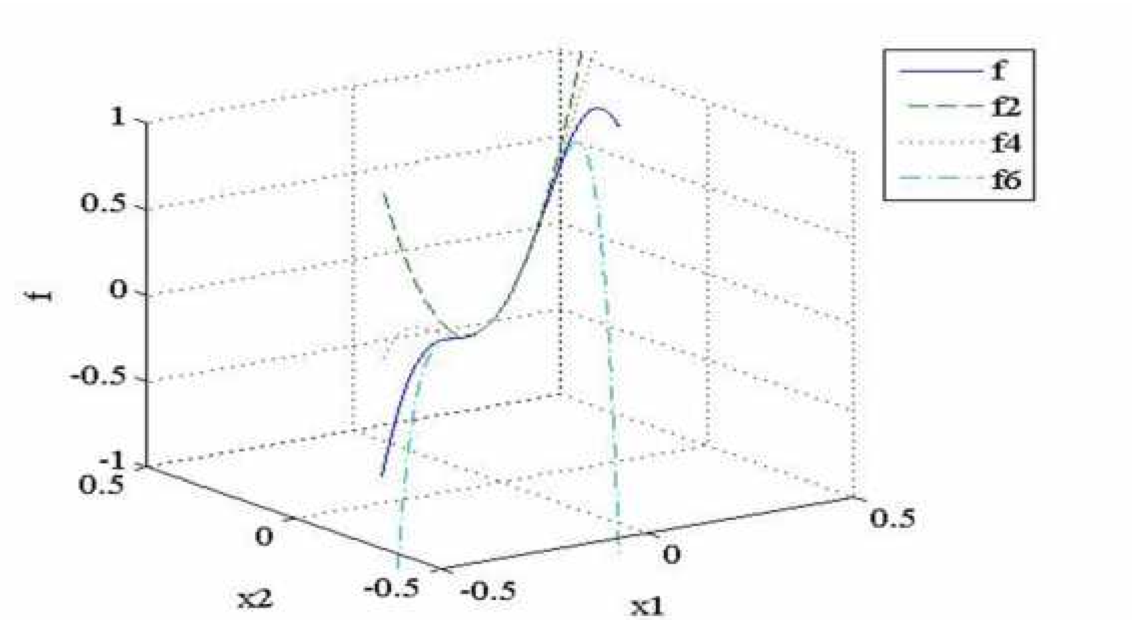

Figure B.2: Graphic representation of f and its polynomial approximations f_2 , f_4 and f_6

The Figure B.2 shows that as long as the order of truncation is higher, the plot of the polynomial approximation function f_i is closer to the plot of the function f .

Table B.2: Radius of best fitting approximation of f *vs.* the order of truncation

Order of truncation	Radius of best fitting approximation (w. $\pm 2\%$)
1	0
2	0.18
3	0.18
4	0.32
5	0.32
6	0.47

The Table B.2 shows the radius of best fitting approximation is higher as long as the order of truncation is higher.

B.3 Example 3.3

Consider the two variable artificial muscle dynamics [37]

$$\dot{x}_1 = x_2 \tag{B3.1}$$

$$\dot{x}_2 = -27.1x_1 - 12.6x_2 + 10.9x_1^2 + 1.3x_2^2 - 1.6x_1^3 - 0.04x_2^3 + u \tag{B3.2}$$

where x_1 represents the position of the muscle and x_2 its velocity. In the following, we consider the unforced system (*i.e.* $u = 0$). Note that the dynamics of the second state variable x_2 is a polynomial of order 3, in the intermediate variables x_1 and x_2. We denote by $f(x_1, x_2)$ the function defining this dynamics

$$f(x_1, x_2) = -27.1x_1 - 12.6x_2 + 10.9x_1^2 + 1.3x_2^2 - 1.6x_1^3 - 0.04x_2^3 \tag{B3.3}$$

We denote also by f_1 and f_2, the linear and second order approximations of f given by

$$f_1\left(x_1, x_2\right) = -27.1x_1 - 12.6x_2 \quad \text{(B3.4)}$$

$$f_2\left(x_1, x_2\right) = -27.1x_1 - 12.6x_2 + 10.9x_1^2 + 1.3x_2^2 \quad \text{(B3.5)}$$

In order to visualize and compare the approximation functions to the original one, we show the phase portrait $x_2 = f\left(x_1\right)$ of the unforced dynamics associated with f, f_1 and f_2, for different initial conditions $\left(x_{10}, x_{20}\right) = (1,0), (-1,0), (0,3)$ and $(0,-3)$. Obviously, the truncated approximation of order 2 is closer than the linerized approximation (see Figure B.3).

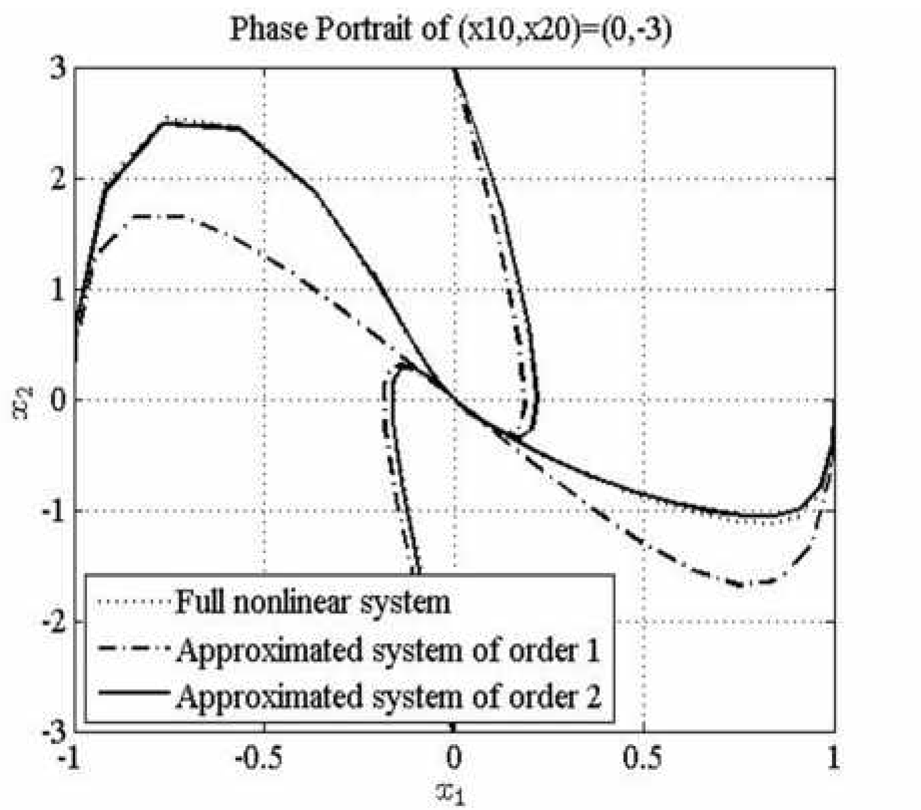

Figure B.3: Phase portrait of full nonlinear dynamics and its approximations of 1st and 2nd order

We can obviously note that the second order approximation has a better fitting curve than the linear approximation to the exact function in terms of domain attraction.

Table B.3 Initial condition *vs.* worst error of approximation of x_2 dynamics in m/s^2

I_c		Estimate Error	
x_{10}	x_{20}	Trunc.1	Trunc.2
1	0	0.57	0.08
-1	0	0.9	0.07
0	3	0.24	0.02
0	-3	0.21	0.01

The Table B.3 shows that for different initial conditions, we calculate the error between the nonlinear system and the polynomial approximation of orders of truncation 1 and 2. We can conclude that the approximation of 2nd order has a lower margin of error than the 1st order, and then we can note that the 2nd order polynomial is a better approximation than the 1st order one. Hence, the second order approximation has a better fitting approximation and a larger domain of approximation than the linear one.

C Illustrative examples and proofs of the optimal control theory

C.1 Example 4.1

Consider the functional

$$J = \int_0^1 \left(x^2 + 2tx + \dot{x}^2 \right) dt \tag{C1.1}$$

with $x(0) = 0$ and $x(1) = 1$. $\dot{x}$ (*resp.* $\ddot{x}$) denotes the first (*resp.* second) derivative of x with respect to t. From (4.17), we obtain

$$\ddot{x} - x - t = 0 \tag{C1.2}$$

A solution of (4.25) can be written

$$x(t) = c_1 e^t + c_1 e^{-t} - t \tag{C1.3}$$

where c_1 and c_2 are constant. Using the boundary conditions $x(0) = 0$ and $x(1) = 1$, we deduce

$$u(t) = \frac{e^t}{\sinh t} - \frac{e^{-t}}{\sinh t} - t \tag{C1.4}$$

C.2 Example 4.2

Consider the minimization problem of the functional J of a single mass-spring (m,k) from classical mechanics. Given m and k the mass and stiffness coefficient. For the position y and its velocity $\dot{y} = \frac{dy}{dt}$ at $t \in [t_1, t_2]$, we propose

$$J = \int_{t_1}^{t_2} (T - U)\, dt \tag{C2.1}$$

where $T = \frac{1}{2} m\dot{y}^2$ is the kinetic energy and $U = \frac{1}{2} ky^2 - mgy$ the potential energy with g is the gravitational acceleration. From the functional minimization problem

$$J = \int_{t_1}^{t_2} \left(\frac{1}{2} m\dot{y} - \frac{1}{2} ky^2 + mgy \right) dt \tag{C2.2}$$

by using (4.16), we obtain the Euler-Lagrange equation

$$m\ddot{y} + ky = mg \tag{C2.3}$$

The trajectory minimizing (C2.2) is

$$y(t) = \frac{mg}{k} + c_1 \sin\left(\sqrt{\frac{k}{m}} t \right) + c_2 \cos\left(\sqrt{\frac{k}{m}} t \right) \tag{C2.4}$$

where c_1 and c_2 are constant. Using the boundary conditions $y(0) = y_0$ and $\dot{y}(0) = 0$, we obtain

$$y(t) = \frac{mg}{k} + \left(y_0 - \frac{mg}{k} \right) \cos\left(\sqrt{\frac{k}{m}} t \right) \tag{C2.5}$$

C.3 Example 4.3

For a volume of a funnel, in the form of right circular cone, estimated at $V_0 = 2000m^3$, the technician wants to construct it from a sheet metal minimizing the lateral surface area. The dimensions of funnel are the radius of the base, r, and the height of the cone, h, respectively. Note that the surface of a right circular cone is $S(r,h) = ® r\sqrt{(r^2+h^2)}$ and its volume $V(r,h) = \frac{®}{3} r^2 h$. The problem is

$$\min\ f(r,h) = S(r,h)^2 \tag{C3.1}$$

subject to

$$g(r,h) = V_0 - V(r,h) \tag{C3.2}$$

The Lagrange function is

$$\begin{aligned} L(r,h,ª) &= f(r,h) + ª g(r,h) \\ &= ®^2 r^2 (r^2+h^2) - \frac{®}{3} ª r^2 h + V_0 ª \end{aligned} \tag{C3.3}$$

The necessary conditions for the solution of the problem are

$$\frac{\partial L}{\partial r} = 4®^2 r^3 + 2®^2 r h^2 - \frac{2®}{3} ª r h = 0 \tag{C3.4}$$

$$\frac{\partial L}{\partial h} = 2®^2 r^2 h - \frac{®}{3} ª r^2 = 0 \tag{C3.5}$$

and

$$\frac{\partial L}{\partial ª} = -\frac{®}{3} r^2 h + V_0 = 0 \tag{C3.6}$$

(C3.5)-(C3.7) lead to

$$h^3 - \frac{h^2}{2} - \frac{3V_0}{®} = 0 \tag{C3.7}$$

$$r^2 = h^2 - \frac{h}{2} \quad \text{(C3.8)}$$

$$\text{ª} = 6\text{®}\, h \quad \text{(C3.9)}$$

The equation (4.58) has one real solution $h_1^* = 12.57$ and two real imaginary solutions $h_2^* = -6.03 + 10.74i$ and $h_2^* = -6.03 - 10.74i$. Since h^* must be real, we have

$$h^* = 12.57 \quad \text{(C3.10)}$$

Then,

$$r^* = 12.31 \quad \text{(C3.11)}$$

and

$$\text{ª}^* = 236.81 \quad \text{(C3.12)}$$

The application of the sufficient condition of (4.36) to (4.38) yields

$$L_{11} = \left.\frac{\partial^2 L}{\partial r^2}\right|_{(r^*, h^*, \text{ª}^*)} = 12\text{®}^2 r^* + 2\text{®}^2 h^{*2} - \frac{2\text{®}}{3}\text{ª}^* h^* = -1596.07 \quad \text{(C3.13)}$$

$$L_{12} = L_{21} = \left.\frac{\partial^2 L}{\partial r \partial h}\right|_{(r^*, h^*, \text{ª}^*)} = 4\text{®}^2 r^* h^* - \frac{2\text{®}}{3}\text{ª}^* r^* = 61.06 \quad \text{(C3.14)}$$

$$L_{22} = \left.\frac{\partial^2 L}{\partial h^2}\right|_{(r^*, h^*, \text{ª}^*)} = 2\text{®}^2 r^{*2} = 2991.79 \quad \text{(C3.15)}$$

$$g_1 = \left.\frac{\partial g}{\partial r}\right|_{(r^*, h^*, \text{ª}^*)} = -\frac{2\text{®}}{3}\text{ª}^* r^* h^* = -75988.22 \quad \text{(C3.16)}$$

$$g_2 = \left.\frac{\partial g}{\partial h}\right|_{(r^*, h^*, \text{ª}^*)} = -\frac{\text{®}}{3}\text{ª}^* r^{*2} = -11856.94 \quad \text{(C3.17)}$$

Then,

$$\begin{vmatrix} L_{11}-z & L_{12} & g_1 \\ L_{21} & L_{22}-z & g_2 \\ g_1 & g_2 & 0 \end{vmatrix} = \begin{vmatrix} -1596.07-z & 61.06 & -75988.22 \\ 61.06 & 2991.79-z & -11856.94 \\ -75988.22 & -11856.94 & 0 \end{vmatrix} = 0 \quad \text{(C3.18)}$$

that is,

$$-1.69\cdot 10^{13} + 5.91\cdot 10^{9} z = 0 \quad \text{(C3.19)}$$

The root of (C3.19) is

$$z = 2859.56 > 0 \quad \text{(C3.20)}$$

Thus, the dimensions $r^* = 12.31$ and $h^* = 12.57$ correspond to a minimum lateral surface area of the funnel of $S^* = 680$.

C.4 Example 4.4

Consider the simple mechanical system composed of a mass-spring-damper given by the dynamic model

$$m\ddot{x} + b\dot{x} + kx = u \quad \text{(C4.1)}$$

where x is the position in m, $v = \dot{x}$ the velocity in m/s, $\ddot{x}$ the acceleration and u the force in N. The mass is $m = 1Kg$, the damping coefficient $b = 0.5\ N.s/m$ and the stiffness $k = 2\ N/m$. The objective to be minimized is

$$J(u) = \int_0^T u^2 dt \quad \text{(C4.2)}$$

which corresponds to the energy consumption.

We determine the optimal control law minimizing the cost functional (C4.2) over the time $[0,T]$, with the final time T specified. This control moves the mass from rest, *i.e.*, $v(0) = 0\ m/s$ to the desired speed v_r at T, *i.e.*, $v(T) = v_r$.

We express (C4.1) in terms of the measured speed as follows

$$m\dot{v} + bv + k\int vdt = u \tag{C4.3}$$

which leads to

$$m\ddot{v} + b\dot{v} + kv = \dot{u} \tag{C4.4}$$

The state-space representation, of the velocity dynamics (C4.4), can be written using the modal representation

$$\dot{x}_1 = a_{12}x_2 + u \tag{C4.5}$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + u \tag{C4.6}$$

and

$$v = c_1x_1 + c_2x_2 \tag{C4.7}$$

with the numerical values $a_{12} = -1.686$, $a_{21} = 1.186$, $a_{22} = -0.5$, $c_1 = 0.25$, $c_2 = 0.25$. That is,

$$\begin{cases} \dot{x} = Ax + Bu \\ v = Cx \end{cases} \tag{C4.8}$$

with

$$A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , \quad C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{C4.9}$$

The details of this modelization are shown as follows. Set the following state space from

$$\dot{x}_1 = a_{12}x_2 + u \tag{C4.10}$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + u \tag{C4.11}$$

and

$$v = c_1 x_1 + c_2 x_2 \tag{C4.12}$$

We have

$$\dot{v} = c_1 \dot{x}_1 + c_2 \dot{x}_2$$

$$= c_1 \left(a_{22} x_2 + u \right) + c_2 \left(a_{21} x_1 + a_{22} x_2 + u \right)$$

$$= c_2 a_{21} x_1 + \left(c_1 a_{12} + c_2 a_{22} \right) x_2 + \left(c_1 + c_2 \right) u \tag{C4.13}$$

and

$$\ddot{v} = c_2 a_{21} \dot{x}_1 + \left(c_1 a_{12} + c_2 a_{22} \right) \dot{x}_2 + \left(c_1 + c_2 \right) \dot{u}$$

$$= c_2 a_{21} \left(a_{12} x_2 + u \right) + \left(c_1 a_{12} + c_2 a_{22} \right) \left(a_{21} x_1 + a_{22} x_2 + u \right) + \left(c_1 + c_2 \right) \dot{u}$$

$$= \left(c_1 a_{12} + c_2 a_{22} \right) a_{11} x_1 + \left(c_2 a_{21} a_{12} + c_2 a_{12} a_{22} + c_2 a_{22}^2 \right) x_2 + \left(c_2 a_{21} + c_1 a_{12} + c_2 a_{22} \right) u + \left(c_1 + c_2 \right) \dot{u} \tag{C4.14}$$

Substitute (C4.10), (C4.11) and (C4.12) into (C4.14)

$$\begin{aligned} &\left(c_1 a_{12} + c_2 a_{22} \right) a_{21} m x_1 + \left(c_2 a_{21} a_{12} + c_2 a_{12} a_{22} + c_2 a_{22}^2 \right) m x_2 + \left(c_2 a_{21} + c_1 a_{12} + c_2 a_{22} \right) m u \\ &+ \left(c_1 + c_2 \right) m \dot{u} + b c_2 a_{21} x_1 + \left(c_1 a_{12} + c_2 a_{22} \right) b x_2 + \left(c_1 + c_2 \right) b u + k c_1 x_1 + k c_2 x_2 = \dot{u} \end{aligned} \tag{C4.15}$$

which is equivalent to

$$\begin{aligned} &\left[\left(c_1 a_{12} + c_2 a_{22} \right) a_{21} m + b c_2 a_{21} + k c_1 \right] x_1 + \left[\left(c_2 a_{21} a_{12} + c_2 a_{12} a_{22} + c_2 a_{22}^2 \right) m \right. \\ &\left. + \left(c_1 a_{12} + c_2 a_{22} \right) b + k c_2 \right] x_2 + \left[\left(c_2 a_{21} + c_1 a_{12} + c_2 a_{22} \right) m + \left(c_1 + c_2 \right) b \right] u + \left[\left(c_1 + c_2 \right) m - 1 \right] \dot{u} = 0 \end{aligned} \tag{C4.16}$$

(C4.16) holds for all x_1, x_2, u and $\dot{u}$. By cancelling the terms of these variables, we obtain

$$\left(c_1 a_{12} + c_2 a_{22} \right) a_{21} m + b c_2 a_{21} + k c_1 = 0 \tag{C4.17}$$

$$\left(c_2 a_{21} a_{12} + c_2 a_{12} a_{22} + c_2 a_{22}^2 \right) m + \left(c_1 a_{12} + c_2 a_{22} \right) b + k c_2 = 0 \tag{C4.18}$$

$$\left(c_2 a_{21} + c_1 a_{12} + c_2 a_{22}\right)m + \left(c_1 + c_2\right)b = 0 \tag{C4.19}$$

and

$$\left(c_1 + c_2\right)m - 1 = 0 \tag{C4.20}$$

Two possible solutions can be obtained

$$a_{12} = -x + \frac{b}{m} \quad ,a_{21} = x \quad ,a_{22} = -\frac{b}{m} \quad and\ c_1 = c_2 = \frac{1}{2m} \text{ with } x = \frac{-b \pm \sqrt{b^2 + 4mk}}{2m}.$$

The Hamiltonian function H is written

$$H\left(x,u,\mathrm{a}\right) = u^2 + \mathrm{a}^T\left(Ax + Bu\right)$$

$$= u^2 + \mathrm{a}_1\left(a_{12}x_2 + u\right) + \mathrm{a}_2\left(a_{21}x_1 + a_{22}x_2 + u\right) \tag{C4.21}$$

We obtain

$$\dot{\mathrm{a}}_1 = -\frac{\partial H}{\partial x_1} = -a_{21}\mathrm{a}_2 \tag{C4.22}$$

$$\dot{\mathrm{a}}_2 = -\frac{\partial H}{\partial x_2} = -a_{12}\mathrm{a}_1 - a_{22}\mathrm{a}_2 \tag{C4.23}$$

and

$$\frac{\partial H}{\partial u} = 2u + \left(\mathrm{a}_1 + \mathrm{a}_2\right) = 0 \tag{C4.24}$$

(C4.22) and (C4.23) are re written as

$$\dot{\mathrm{a}}\left(t\right) = \begin{pmatrix} 0 & -a_{21} \\ -a_{12} & -a_{22} \end{pmatrix}\mathrm{a} = -A^T\mathrm{a}\left(t\right) \tag{C4.25}$$

Then, we obtain

$$\mathrm{a}\left(t\right) = e^{-A^T t}c_0 \tag{C4.26}$$

with $c_0 \in \mathbb{R}^2$ a constant vector. From (C4.24), we have

$$u(t) = -\frac{1}{2}(\lambda_1 + \lambda_2)$$

$$= -\frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\lambda(t)$$

$$= -\frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}e^{-A^T t}c_0 \qquad \text{(C4.27)}$$

The transversality conditions at T specified are written from (4.93)

$$\lambda(T)^T \cdot \delta x(T) = \lambda_1(T)\cdot\delta x_1(T) + \lambda_2(T)\cdot\delta x_2(T) = 0 \qquad \text{(C4.28)}$$

As $\delta T = 0$ (T is specified). Noting $v(T) = c_1 x_1(T) + c_2 x_2(T) = c_1\left(x_1(T) + x_2(T)\right)$ is known as $c_1 = c_2$, then $\delta x_1(T) + \delta x_2(T) = 0$. So, for $\delta x_1(T) \neq 0$,

$$\left(\lambda_1(T) - \lambda_2(T)\right)\cdot\delta x_1(T) = 0 \qquad \text{(C4.29)}$$

We obtain $\lambda_1(T) = \lambda_2(T) = \lambda_T$.

From (4.121), we have $\lambda(T) = e^{-A^T T}c_0 = \begin{pmatrix}1\\1\end{pmatrix}\lambda_T$, *i.e.*, $c_0 = e^{A^T}\begin{pmatrix}1\\1\end{pmatrix}\lambda_T$. Thus,

$$u(t) = -\frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}e^{-A^T t}e^{A^T t}\begin{pmatrix}1\\1\end{pmatrix}\lambda_T$$

$$= -\frac{1}{2}B^T e^{A^T(T-t)}B\lambda_T \qquad \text{(C4.30)}$$

$$u(t) = e^{-0.25(T-t)}\cos\left(1.392(T-t)\right)\lambda_T - 0.180e^{-0.25(T-t)}\sin\left(1.392(T-t)\right)\lambda_T \qquad \text{(C4.31)}$$

Integrating (C4.31) between 0 and T, and using (C4.29) and $x_1(0) = x_2(0) = 0$, we obtain

$$x(T) = -\frac{a_T}{2}\int_0^T e^{A(T-t)}BB^T e^{A^T(T-t)}B\,dt \qquad \text{(C4.32)}$$

and, $v(T) = C \cdot x(t) = v_r$. Given $T = 2$ and $v_c = 1\,m/s$, we obtain

$$a_T = 1.707 \text{ and } v_r = 1.707 \qquad \text{(C4.33)}$$

The optimal cost is $J^* = 54.36$. The results are shown in Figures C1. and C2. The obtained optimal control allows to reach the speed $v_r = 1\,m/s$ within $T = 2\,s$.

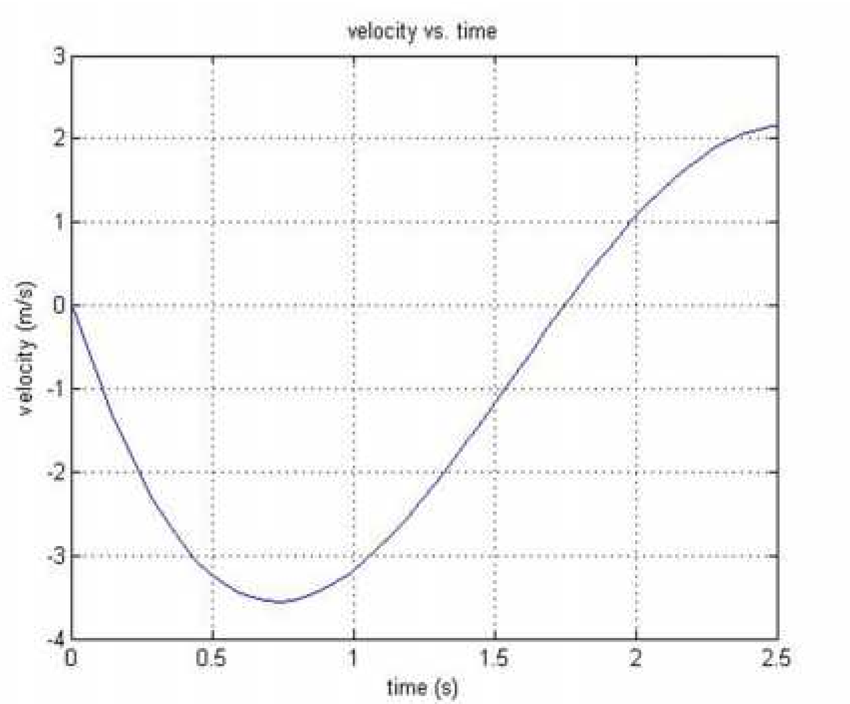

Figure C1. Velocity evolutions *vs.* time of the mass spring damper system

Figure C2. Force evolutions *vs.* time of the mass spring damper system

C.5 Example 4.5

Consider the system [44]

$$\dot{x} = e^x u \tag{C5.1}$$

with the performance objective

$$J = \frac{1}{2}\int_0^\infty \left(x^2 + u^2\right) dt \tag{C5.2}$$

The HJE problem is stated as

$$\frac{1}{2}\left(x^2 + u^2\right) - \frac{\partial V^2}{\partial x} e^x u = 0 \tag{C5.3}$$

The optimal control is given by

$$u^* = -\frac{\partial V}{\partial x} e^x \qquad \text{(C5.4)}$$

Substituting (C5.4) into (C5.3) leads to

$$\frac{\partial V}{\partial x} = x e^{-x} \qquad \text{(C5.5)}$$

Then,

$$u^* = -x \qquad \text{(C5.6)}$$

The optimal cost is estimated at $T^* = 0.264$, the state evolution is shown in Figure C3.

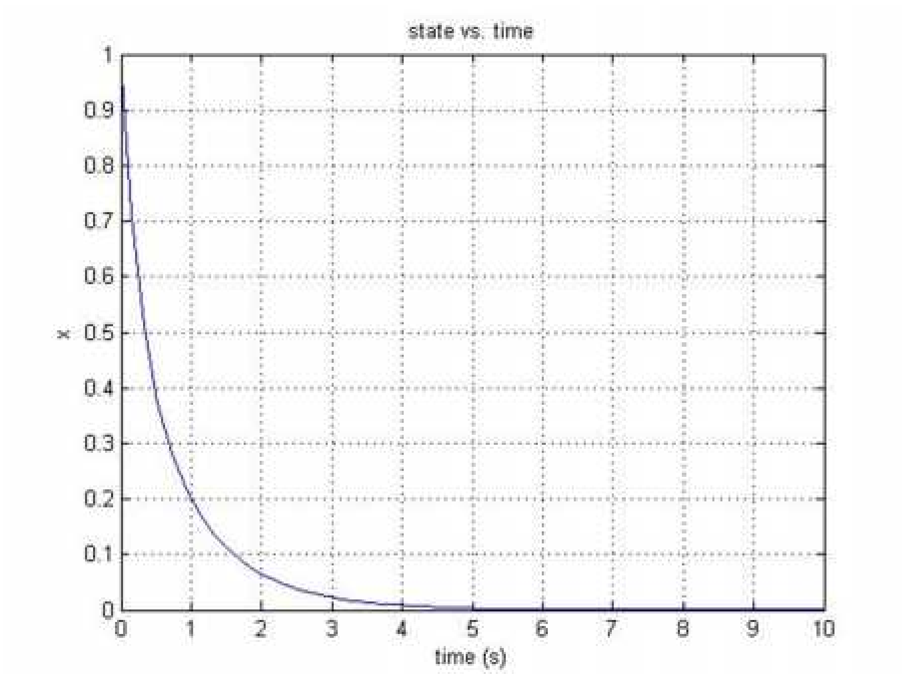

Figure C3. State evolutions *vs.* time of the scalar example (4.159)

C.6 Proof of Theorem 4.2

The optimality condition

$$\min_u \left(\frac{dV}{dt} + x^T Qx + u^T Ru + 2x^T Nu \right) = 0 \tag{C6.1}$$

can be rewritten

$$\left.\frac{dV}{dt}\right|_{u=u^*} + x^T Qx + u^{*T} Ru^* + 2x^T Nu = 0 \tag{C6.2}$$

that is,

$$\left.\frac{dV}{dt}\right|_{u=u^*} = -x^T Qx - u^{*T} Ru^* - 2x^T Nu^* \tag{C6.3}$$

Integrating both sides of the resulting equation with respect to time from 0 to ∞, we obtain

$$V\left(x(\infty)\right) - V\left(x(0)\right) = -\int_0^\infty \left(x^T Qx + u^{*T} Ru^* + 2x^T Nu \right) dt \tag{C6.4}$$

Since we assume that the closed loop system is asymptotically stable, we have $x(\infty) = 0$ and $V\left(x(\infty)\right) = 0$. Then, we obtain

$$V\left(x(0)\right) = x_0^T Px_0 = \int_0^\infty \left(x^T Qx + u^{*T} Ru^* + 2x^T Nu^* \right) dt \tag{C6.5}$$

Thus, the value of the performance index for such a stabilizing controller is

$$J\left(u^*\right) = x_0^T Px_0 \tag{C6.6}$$

To show that such a controller is optimal, we use a proof by contradiction. We assume that (C6.1) holds and that u^* is not optimal. Suppose that a control $\tilde{u}$ yields a smaller value of J, that is,

$$J\left(\tilde{u}\right) < J\left(u^*\right) \tag{C6.7}$$

It follows from (C6.1) that

$$\left.\frac{dV}{dt}\right|_{u=\tilde{u}} + x^T Q x + \tilde{u}^T R \tilde{u} + 2x^T N \tilde{u} \geq 0 \tag{C6.8}$$

that is

$$\left.\frac{dV}{dt}\right|_{u=\tilde{u}} \geq -x^T Q x - \tilde{u}^T R \tilde{u} - 2x^T N \tilde{u} \tag{C6.9}$$

Integrating (C6.9) with respect to time from 0 to ∞, we obtain

$$V\left(x(0)\right) \leq \int_0^\infty \left(x^T Q x + \tilde{u}^T R \tilde{u} + 2x^T N \tilde{u}\right) dt \tag{C6.10}$$

which implies that

$$J\left(u^*\right) = x_0^T P x_0 \leq J\left(\tilde{u}\right) \tag{C6.11}$$

(C6.11) is in contradiction with (C6.7). Hence u^* is optimal.

C.7 Proof of Theorem 4.6

Consider the following equation

$$\left(A - BR^{-1}N^T\right)^T P + P^T\left(A - BR^{-1}N^T\right) + \left(Q - NR^{-1}N^T\right) - P^T BR^{-1}B^T P = 0 \tag{C7.1}$$

Denote by $\bar{A} = A - BR^{-1}N^T$ and $\bar{Q} = Q - NR^{-1}N^T$, the ARE (4.177) is written

$$\bar{A}^T P + P^T \bar{A} + \bar{Q} - P^T BR^{-1}B^T P = 0 \tag{C7.2}$$

Let we prove that if $\bar{Q} = Q - NR^{-1}N^T > 0$, then $P > 0$. First, let we prove that $Ker(P) \subseteq Ker(Q)$.

Assume $x \in Ker(P)$, *i.e.*, $P \cdot x = 0$. From (4.178), we have

$$x^T \bar{A}^T Px + x^T P^T \bar{A} + x^T Qx - x^T P^T BR^{-1} B^T Px = 0 \tag{C7.3}$$

which implies $x^T \bar{Q} x = 0$, then $\bar{Q} x = 0$. Then, if $\bar{Q} > 0$ that is $x\bar{Q}x = 0$, then $x = 0$. Thus, $Ker(\bar{Q}) = \{0\}$. So, $Ker(P) \subseteq \{0\}$. Then, $Ker(P) = \{0\} \Rightarrow P > 0$.

Now, we prove if $\bar{Q} = Q - NR^{-1}N^T \geq 0$, then $P > 0$. Recall $\bar{A} = A - BR^{-1}N^T$ and define $A_c = \bar{A} - BR^{-1}B^T P$. Then, (C7.3) can be written as

$$P^T \left(\bar{A} - BR^{-1}B^T P\right) + \left(\bar{A} - BR^{-1}B^T P\right)^T P + \bar{Q} + P^T BR^{-1}B^T P = 0 \tag{C7.4}$$

that is,

$$P^T A_c + A_c^T P + \bar{Q} + P^T BR^{-1}B^T P = 0 \tag{C7.5}$$

Pre-multiply and post-multiply (C7.5) by $z^T e^{A_c^T t}$ and $e^{A_c t} z$, respectively

$$\begin{aligned} &z^T e^{A_c^T t} P^T A_c e^{A_c t} z + z^T e^{A_c^T t} A_c^T P e^{A_c t} z + z^T e^{A_c^T t} \bar{Q} e^{A_c t} z \\ &+ z^T e^{A_c^T t} P^T BR^{-1} B^T P e^{A_c t} z = 0 \end{aligned} \tag{C7.6}$$

Let we calculate the following time derivative

$$\begin{aligned} \frac{d}{dt}\left[z^T e^{A_c^T t} P^T e^{A_c t} z\right] &= \frac{d}{dt}\left[\left(e^{A_c t} z\right)^T P^T \left(e^{A_c t} z\right)\right] \\ &= 2\left(e^{A_c t} z\right)^T P^T \frac{d}{dt}\left(e^{A_c t} z\right) \\ &= 2\left(e^{A_c t} z\right)^T P^T A_c e^{A_c t} z \\ &= 2 z^T e^{A_c^T t} P^T A_c e^{A_c t} z \\ &= z^T e^{A_c^T t} P^T A_c e^{A_c^T t} z + z^T e^{A_c^T t} A_c^T P e^{A_c t} z \end{aligned} \tag{C7.7}$$

Considering $R > 0$, *i.e.*, $R^{-1} > 0$. Then, using the Cholesky decomposition [50], $\exists \bar{R} > 0$ such that $R^{-1} = \bar{R}^T \bar{R}$. We write

$$z^T e^{A_c^T t} P^T B R^{-1} B^T P e^{A_c t} z = z^T e^{A_c^T t} P^T B \bar{R}^T \bar{R} B^T P e^{A_c t} z$$
$$= \left(\bar{R} B^T P e^{A_c t} z\right)^T \left(\bar{R} B^T P e^{A_c t} z\right)$$
$$= \left\| \bar{R} B^T P e^{A_c t} z \right\|^2 \qquad \text{(C7.8)}$$

Then, combining (C7.6), (C7.7) and (C7.8), we obtain

$$\frac{d}{dt}\left[z^T e^{A_c^T t} P^T e^{A_c^T t} z \right] = -z^T e^{A_c^T t} \bar{Q} e^{A_c t} z - \left\| \bar{R} B^T P e^{A_c t} z \right\|^2 \qquad \text{(C7.9)}$$

Integrating (C7.9) from 0 to t, we have

$$\int_0^t \frac{d}{d\tau}\left[z^T e^{A_c^T \tau} P^T e^{A_c \tau} z \right] d\tau = -\int_0^t \left(z^T e^{A_c^T \tau} \bar{Q} e^{A_c \tau} z + \left\| \bar{R} B^T P e^{A_c \tau} z \right\|^2 \right) d\tau \qquad \text{(C7.10)}$$

that is,

$$\left(z^T e^{A_c^T \tau} P^T e^{A_c \tau} z \right)\Big|_0^T = -\int_0^t \left(z^T e^{A_c^T \tau} \bar{Q} e^{A_c \tau} z + \left\| \bar{R} B^T P e^{A_c \tau} z \right\|^2 \right) d\tau \qquad \text{(C7.11)}$$

or equivalently,

$$z^T e^{A_c^T t} P^T e^{A_c t} z - z^T P^T z = -\int_0^t \left(z^T e^{A_c^T \tau} \bar{Q} e^{A_c \tau} z + \left\| \bar{R} B^T P e^{A_c \tau} z \right\|^2 \right) d\tau \qquad \text{(C7.12)}$$

Since the integrant term, the left side of the equality (C7.12), is non-negative. Then,

$$0 \le z^T e^{A_c^T t} P^T e^{A_c t} z \le z^T P^T z = z^T P z \qquad \text{(C7.13)}$$

Note that if $\exists z \neq 0$ such that $Pz = 0$, then

$$P e^{A_c t} z = 0 \qquad \text{(C7.14)}$$

Hence, we have $\forall z \in Ker\left(P\right)$, *i.e.*, $Pz = 0$. So, for any given $z \in Ker\left\{P\right\}$, we have

$$A_c z = \left(\bar{A} - BR^{-1}B^T P\right)z = \bar{A}z - BR^{-1}B^T Pz = \bar{A}z \tag{C7.15}$$

Then, $\forall t \geq 0$

$$e^{A_c t} z = e^{\bar{A}t} z \tag{C7.16}$$

In fact, note that for any matrix M, we define the exponential matrix of M, denoted by e^M, as $e^M = \sum_{i \geq 0} \frac{1}{i!} M^i$. From (4.246), we have $\forall i \geq 1$ and $\forall z \in Ker\left(P\right)$,

$$\left(A_i - \bar{A}\right)^i z = 0 \tag{C7.17}$$

Then, $\forall z \in Ker\left(P\right)$ and $\forall t \geq 0$

$$\begin{aligned} e^{\left(A_i - \bar{A}\right)t} z &= \sum_{i \geq 0} \frac{t^i}{i!} \left(A_c - \bar{A}\right)^i z \\ &= z + \sum_{i \geq 1} \frac{t^i}{i!} \left(A_c - \bar{A}\right)^{i-1} \left(A_c - \bar{A}\right) z \\ &= z \end{aligned} \tag{C7.18}$$

We write

$$\begin{aligned} \left(e^{A_c t} - e^{\bar{A}t}\right) z &= e^{\bar{A}t} \left(e^{-\bar{A}t} e^{A_c t} - I\right) z \\ &= e^{\bar{A}t} \left(e^{\left(A_c - \bar{A}\right)t} - I\right) z \\ &= e^{\bar{A}t} \left(e^{\left(A_c - \bar{A}\right)t} z - z\right) \\ &= e^{\bar{A}t} \left(z - z\right) \\ &= 0 \end{aligned} \tag{C7.19}$$

Thus, from (C7.18), we obtain (C7.19). Now, pre-multiply (C7.16) by P

$$Pe^{A_c t}z = Pe^{\bar{A}t}z \tag{C7.20}$$

From (C7.14), we obtain $Pe^{\bar{A}t}z = 0$, *i.e.*, $Pe^{\bar{A}t}z \in Ker(P)$. Thus, $Ker(P)$ is invariant under $e^{\bar{A}t}$. Also, by pre-multiplying and post-multiplying (C7.5) by z^T and z respectively, we obtain $\forall z \in Ker(P)$, $\bar{Q}z = 0$. Then, $Ker(P) \subseteq Ker(\bar{Q})$. Assuming $Pe^{A_c t}z = 0$, we obtain, from (4.236), $\bar{Q}e^{A_c t}z = 0$ with $z \in Ker(P)$. Then,

$$\bar{Q}e^{A_c t}z = \bar{Q}e^{\bar{A}t} = 0 \tag{C7.21}$$

(C7.21) represents a contradiction to $(\bar{A}, \bar{Q})$ observable, according to Theorem 4.6. In fact, as $z \neq 0$, then $Ker(\bar{Q}e^{\bar{A}t}) \neq \{0\}$ and equivalently $(\bar{A}, \bar{Q})$ is unobservable; which is a contradiction.

D 2-DOF helicopter set-up kinetic and dynamic models

D.1 Coordinates transformations

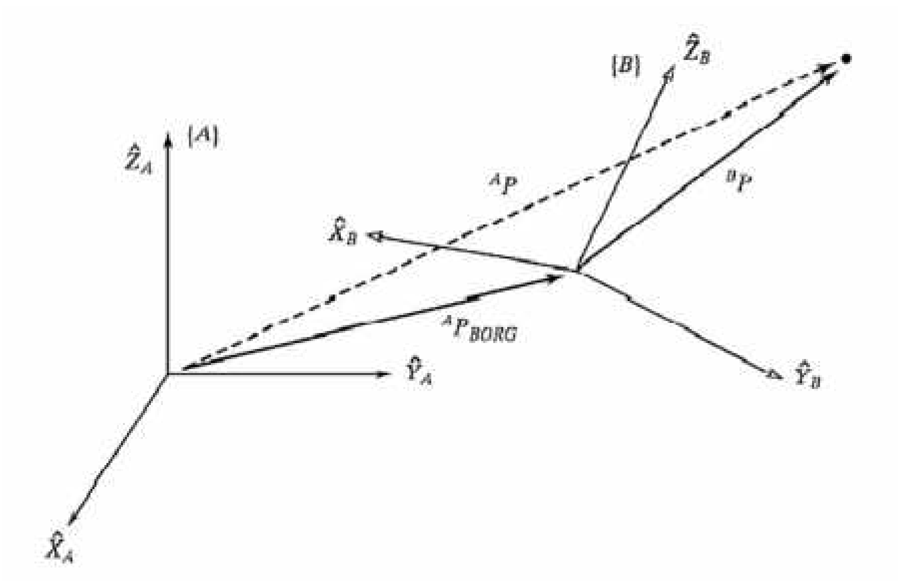

Fig D.1 General Transformation of a vector [62]

Very often we know the description of a vector with respect to some frame, $\{B\}$, and we would like to know its description with respect t another frame, $\{A\}$. In the general case of mapping the origin of the frame $\{B\}$ is not coincident with that of frame $\{A\}$ but has a general vector offset. The vector that locates $\{B\}$'s origin is called $^{A}P_{Borg}$. Also $\{B\}$ is rotated with respect to $\{A\}$ as described by $^{A}_{B}R$. Given ^{B}P, we wish to compute ^{A}P as in Fig.A.1. We can first change ^{B}P to its description relative to an intermediate frame which has some orientation as $\{A\}$, but whose origin is coincident with the origin of $\{B\}$. This is done by multiplying by ${}^{A}_{B}R$, then we account for the translation between origins by simple vector addition yielding [62]

$$ {}^{A}P = {}^{A}_{B}R\ {}^{B}P + {}^{A}P_{Borg} \tag{D1.1} $$

Equation (D1.1) describes a general transformation mapping of a vector from its description in one frame to its description in a second frame. It can be written in a compact form as

$$ {}^{A}P = {}^{A}_{B}T\ {}^{B}P \tag{D1.2} $$

Where ${}^{A}_{B}T$ is called the homogenous transformation matrix. It can be written in a (4x4) matrix as follows

$$ {}^{A}_{B}T = \begin{bmatrix} & & & \vdots & \\ & {}^{A}_{B}R & & \vdots & {}^{A}P_{Borg} \\ & & & \vdots & \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix} \tag{D1.3} $$

where ${}^{A}_{B}R$ is a (3x3) matrix which represents the rotational component of the transformation matrix and ${}^{A}P_{Borg}$ is a (3x1) matrix which represents the translational component.

D.2 Kinematic model of the 2-DOF helicopter

As illustrated in Fig D.2, we define the following coordinates systems:

$O_3x_3y_3z_3$ is the frame located at the center of mass of the helicopter. It is related to the frame $Ox_2y_2z_2$ by a translation of a distance l_{cm} in the direction of the axis x_2.

$Ox_2y_2z_2$ is the frame located at the center of the front propeller. It is related to the frame $Ox_1y_1z_1$ by a rotation of an angle ¯ around the axis y_1.

$Ox_1y_1z_1$ is the frame located at the center of the back propeller. It is related to the frame $Ox_0y_0z_0$ by a rotation of an angle · around the axis z_0.

$Ox_0y_0z_0$ is the frame located at the pivot point of the helicopter which is defined as the point where the pitch axis and yaw axis intersect, or the midpoint of the pitch axis

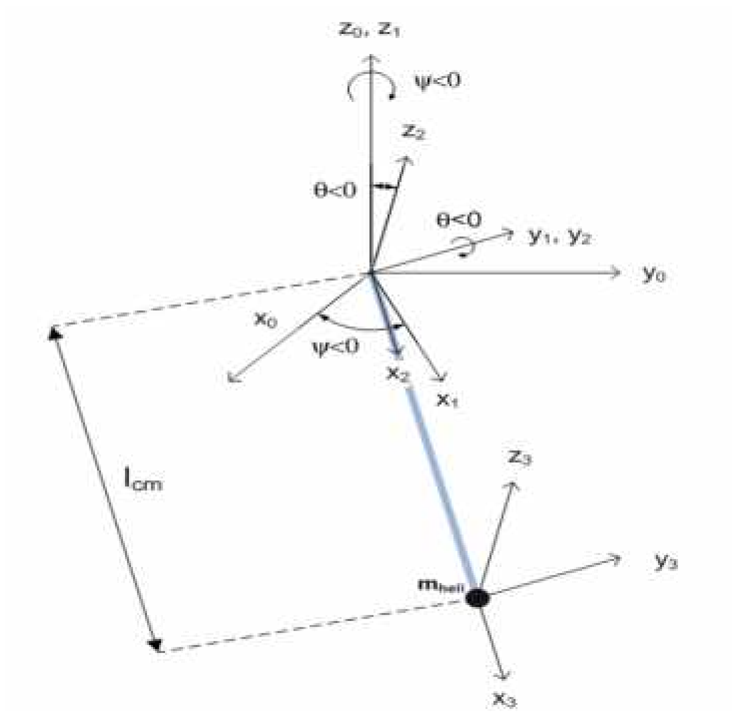

Fig D.2 Kinematic model of the 2-DOF helicopter [61]

Let $[x_2 \quad y_2 \quad z_2]^T$ the coordinates of the center of mass O_3 in the frame $Ox_2y_2z_2$, according to (A.2) it can be written as

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}_{Ox_2y_2z_2} = \begin{bmatrix} 1 & 0 & 0 & l_{cm} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}_{O_3x_3y_3z_3} \tag{D2.1}$$

Let $[x_1 \quad y_1 \quad z_1]^T$ the coordinates of the center of mass O_3 in the frame $Ox_1y_1z_1$, according to (D1.2) it can be written as

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}_{Ox_1y_1z_1} = \begin{bmatrix} \cos{}^{-} & \sin{}^{-} & 0 & 0 \\ -\sin{}^{-} & \cos{}^{-} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}_{Ox_2y_2z_2} \qquad \text{(D2.2)}$$

Let $[x_0 \quad y_0 \quad z_0]^T$ the coordinates of the center of mass O_3 in the frame $Ox_1y_1z_1$, according to (D1.2) it can be written as

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}_{Ox_0y_0z_0} = \begin{bmatrix} \cos\cdot & \sin\cdot & 0 & 0 \\ -\sin\cdot & \cos\cdot & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}_{Ox_1y_1z_1} \qquad \text{(D2.3)}$$

By replacing (D2.1) into (D2.2) and (D2.2) into (D2.3), the latter will be

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}_{Ox_0y_0z_0} = \begin{bmatrix} \cos\cdot & \sin\cdot & 0 & 0 \\ -\sin\cdot & \cos\cdot & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos{}^{-} & \sin{}^{-} & 0 & 0 \\ -\sin{}^{-} & \cos{}^{-} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & l_{cm} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}_{O_3x_3y_3z_3} \qquad \text{(D2.4)}$$

In a compact form, the equation (D2.4) will be

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}_{Ox_0y_0z_0} = \begin{bmatrix} \cos\cdot\ \cos{}^{-} & \sin\cdot & -\cos\cdot\ \sin{}^{-} & l_{cm}\cos\cdot\ \cos{}^{-} \\ -\sin\cdot\ \cos{}^{-} & \cos\cdot & \sin\cdot\ \sin{}^{-} & -l_{cm}\sin\cdot\ \cos{}^{-} \\ \sin{}^{-} & 0 & \cos{}^{-} & l_{cm}\sin{}^{-} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}_{O_3x_3y_3z_3} \qquad \text{(D2.5)}$$

Hence

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}_{Ox_0y_0z_0} = \begin{bmatrix} l_{cm}\cos\cdot\ \cos{}^{-} \\ -l_{cm}\sin\cdot\ \cos{}^{-} \\ l_{cm}\sin{}^{-} \\ 1 \end{bmatrix}_{O_3x_3y_3z_3} \qquad \text{(D2.6)}$$

If we note $x_{cm} = x_0$, $y_{cm} = y_0$ and $z_{cm} = z_0$, the equation (D2.6) leads to the cartesian position of the helicopter center of mass is

$$\begin{cases} x_{cm} = l_{cm} \cos \cdot \ \cos \bar{} \\ y_{cm} = -l_{cm} \sin \cdot \ \cos \bar{} \\ z_{cm} = l_{cm} \sin \bar{} \end{cases} \tag{D2.7}$$

D.3 Kinetic and potential energy

The potential energy due to the gravity is

$$\begin{aligned} V &= m_{hel} \cdot g \cdot z_{cm} \\ &= m_{hel} \cdot g \cdot l_{cm} \cdot \sin \bar{} \end{aligned} \tag{D3.1}$$

The total Kinetic energy is

$$T = T_{r,p} + T_{r,y} + T_t \tag{D3.2}$$

The total kinetic energy T is the sum of the rotational kinetic energies acting from the pitch, $T_{r,p}$ and from the yaw $T_{r,y}$ along with the translational kinetic energy generated by the moving center of mass T_t.

The pitch rotational kinetic energy is

$$T_{r,p} = \frac{1}{2} J_{eq,p} \dot{\bar{}}^{\,2} \tag{D3.3}$$

The yaw rotational kinetic energy is

$$T_{r,y} = \frac{1}{2} J_{eq,y} \dot{}^{\,2} \tag{D3.4}$$

Where $J_{eq,p}$ and $J_{eq,y}$ are the equivalent moment of inertias of the pitch and yaw, respectively.

The translational kinetic energy is

$$T_t = \frac{1}{2} m_{hel} \sqrt{\dot{x}_{cm}^2 + \dot{y}_{cm}^2 + \dot{z}_{cm}^2} \tag{D3.5}$$

By deriving the equation (D2.7), the three dimensional velocity of the center of mass is

$$\begin{cases} x_{cm} = -l_{cm}\left(\dot{\cdot}\ \sin\cdot\ \cos\bar{}\ + \dot{\bar{}}\ \cos\cdot\ \sin\bar{}\right) \\ y_{cm} = l_{cm}\left(-\dot{\cdot}\ \cos\cdot\ \cos\bar{}\ + \dot{\bar{}}\ \sin\cdot\ \sin\bar{}\right) \\ z_{cm} = l_{cm}\dot{\bar{}}\ \cos\bar{} \end{cases} \tag{D3.6}$$

In terms of the pitch and yaw angles the translational kinetic energy is

$$T_t = \frac{1}{2} m_{hel}\left(-l_{cm}\dot{\cdot}\ \sin\cdot\ \cos\bar{} - l_{cm}\dot{\bar{}}\cos\cdot\ \sin\bar{}\right)^2 + \left(-l_{cm}\dot{\cdot}\ \cos\cdot\ \cos\bar{} + l_{cm}\dot{\bar{}}\sin\cdot\ \sin\bar{}\right)^2 + l_{cm}^2\dot{\bar{}}^2\cos\bar{}^2 \tag{D3.7}$$

Hence the total kinetic energy of the system is:

$$\begin{aligned} T = &\frac{1}{2}J_{eq,p}\dot{\bar{}}^2 + \frac{1}{2}J_{eq,y}\dot{\cdot}^2 + \frac{1}{2}m_{hel}\left(-l_{cm}\dot{\cdot}\ \sin\cdot\ \cos\bar{} - l_{cm}\dot{\bar{}}\cos\cdot\ \sin\bar{}\right)^2 \\ &+\left(-l_{cm}\dot{\cdot}\ \cos\cdot\ \cos\bar{} + l_{cm}\dot{\bar{}}\sin\cdot\ \sin\bar{}\right)^2 + l_{cm}^2\dot{\bar{}}^2\cos\bar{}^2 \end{aligned} \tag{D3.8}$$

D.4 Equation of motion

The Lagrangian L is the difference between the kinetic and potential energy of the system: $L = T - V$

$$\begin{aligned} L = &\frac{1}{2}J_{eq,p}\dot{\bar{}}^2 + \frac{1}{2}J_{eq,y}\dot{\cdot}^2 + \frac{1}{2}m_{hel}\left(-l_{cm}\dot{\cdot}\ \sin\cdot\ \cos\bar{} - l_{cm}\dot{\bar{}}\cos\cdot\ \sin\bar{}\right)^2 \\ &+\left(-l_{cm}\dot{\cdot}\ \cos\cdot\ \cos\bar{} + l_{cm}\dot{\bar{}}\sin\cdot\ \sin\bar{}\right)^2 + l_{cm}^2\dot{\bar{}}^2\cos\bar{}^2 - m_{hel}\cdot g\cdot l_{cm}\cdot\sin\bar{} \end{aligned} \tag{D4.1}$$

The generalized coordinates are

$$q = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^T = \begin{bmatrix} - & \cdot & \dot{-} & \dot{\cdot} \end{bmatrix}^T \tag{D4.2}$$

The generalized forces are

$$\begin{cases} Q_1 = K_{pp}V_{m,p} + K_{py}V_{m,y} - B_p\dot{-} \\ Q_2 = K_{yp}V_{m,p} + K_{yy}V_{m,y} - B_y\dot{\cdot} \end{cases} \tag{D4.3}$$

The input controls are $V_{m,p}$ is the input pitch motor voltage and $V_{m,y}$ is the input yaw motor voltage. B_p and B_y are the viscous rotary friction acting about the pitch and yaw axis. $K_{pp}, K_{yy}, K_{py}, K_{yp}$ are the thrust force constants acting on pitch/yaw axis from pitch/yaw motor propeller.

The Euler-Lagrange equations are given by

$$\begin{cases} \dfrac{\partial}{\partial t}\dfrac{\partial L}{\partial \dot{q}_1} - \dfrac{\partial L}{\partial q_1} = Q_1 \\ \dfrac{\partial}{\partial t}\dfrac{\partial L}{\partial \dot{q}_2} - \dfrac{\partial L}{\partial q_2} = Q_2 \end{cases} \tag{D4.4}$$

By applying the equations (D4.1) and (D4.3) into the equations (D4.4), leads to the nonlinear equation of motions

$$\begin{cases} \left(J_{eq,p} + m_{hel}l_{cm}^2\right)\ddot{-} = -m_{hel}gl_{cm}\cos - - B_p\dot{-} - m_{hel}l_{cm}^2 \sin - \cos - \dot{\cdot}^2 + K_{pp}V_{m,p} + K_{py}V_{m,y} \\ \left(J_{eq,y} + m_{hel}l_{cm}^2 \cos -^2\right)\ddot{\cdot} = 2m_{hel}l_{cm}^2 \sin - \cos - \dot{\cdot}\dot{-} - B_y\dot{\cdot} + K_{yp}V_{m,p} + K_{yy}V_{m,y} \end{cases} \tag{D4.5}$$

E MATLAB code of calculation of control gain matrices for 2-DOF helicopter-model set-up

```
% Initialization and Matrix Gain Calculus file
% Optimal control using Lyapunov-based Kronecker product tensor

%%
clear all
clc

%% ~~~~~~~~~~~~~~~~~~~~~~~~~~ 2-DOF HELI CONFIGURATION: Model
Parameters %%
% Set the model parameters of the 2-DOF HELI.
% Copyright (C) 2006 Quanser Consulting Inc.

% Parameters for Expeiments
% Cable Gain used for yaw and pitch axes.
K_CABLE_P = 5;
K_CABLE_Y = 3;
% Amplifier/Voltage and Position Settings
% Amplifier Gain: set to 3 when using VoltPAQ-X2.
% NOTE: If using VoltPAQ-X1, make sure both Gain switches are set to 3.
K_AMP = 3;
% Maximum Output Voltage (V): YAW limited to 15 V. PITCH limited to 24 V.
VMAX_AMP_P = 24;
VMAX_AMP_Y = 15;
% Digital-to-Analog Maximum Voltage (V): set to 10 for Q4/Q8 cards
VMAX_DAC = 10;
% Pitch and Yaw Axis Encoder Resolution (rad/count)
K_EC_P = - 2 * pi / ( 4 * 1024 );
K_EC_Y = 2 * pi / ( 8 * 1024 );
% Specifications of a second-order low-pass filter
wcf = 2 * pi * 20; % filter cutting frequency
zetaf = 0.85;       % filter damping ratio

% Gravitational Constant (m/s^2)
g = 9.81;
% Pitch and Yaw Motor Armature Resistance (Ohm)
```

MATLAB code of calculation of control gain matrices for 2-DOF helicopter-model set-up

```
R_m_p = 0.83;
R_m_y = 1.60;
% Pitch and Yaw Motor Current-Torque Constant (N.m/A)
K_t_p = 0.0182;
K_t_y = 0.0109;
% Pitch and Yaw Propeller Torque-Thrust Constant (N.m/V)
K_pp = 0.2041;
K_yy = 0.0270;
% Pitch and Yaw Motor Voltage-Torque Constant (N.m/V)
K_yp = 0.0219;
K_py = 0.0068;
% Pitch and Yaw Viscous Damping Constant (N.m.s/rad)
B_p = 0.8;   % Tuned while running simulation and experiment in parallel
B_y = 0.318; % Identified as described in manual
% Mass of the Helicopter (kg)
m_heli = 1.3872;
% Helicopter Center of Mass from Pivot along Pitch Axis (m)
l_cm = 0.1476;
% Equivalent Moment of Inertia about Pitch and Yaw Axis (kg.m^2)
J_eq_p = 0.0384;
J_eq_y = 0.0432;
%
% UPM Maximum Output Voltage (V): YAW has UPM-15-03 and PITCH has UPM-24-
05
VMAX_UPM_P = 24;
VMAX_UPM_Y = 15;

%% ~~~~~~~~~~~~~~~~~~~~~~~~~ Paramter Inialization for Controller Design %%
% Set the control parameter design.

% Feed-forward gain adjustment (V/V)
K_ff = 1;

% State Vector: X = [ theta; psi; theta_dot; psi_dot]
n  = 4; % Number of States
% Input Vector: U = [ u_Pitch; u_Yaw]
m  = 2; % Number of Inputs

% Operational point
theta_o = 0;

x1o = theta_o;    x2o = 0;    x3o = 0;    x4o = 0;
Xo = [x1o;
     x2o;
     x3o;
     x4o];
```

MATLAB code of calculation of control gain matrices for 2-DOF helicopter-model set-up

```
u1o =  m_heli*g*l_cm*K_yy/(K_pp*K_yy-K_py*K_yp);
u2o = -m_heli*g*l_cm*K_yp/(K_pp*K_yy-K_py*K_yp);
Uo = [u1o;
     u2o];

% Cost functional coefficients
R = eye(m);
Q = diag([200, 200, 100, 100]);

% Truncation Order
nf = 4;% Truncation Order of F(x)
ng = 4;% Truncation Order of G(x)
nh = 4;% Truncation Order of H(x)

% Matrix F1 of F(x)
F1 = [ 0, 0, 1                      ,                   0;
      0, 0,                      0,                     1;
      0, 0,-B_p/(J_eq_p+m_heli*l_cm^2),                   0;
      0, 0,                      0,-B_y/(J_eq_y+m_heli*l_cm^2)];

% Matrix F2 of F(x)
F2 = zeros(n,n^2);
F2(3, 1) = 1/2*m_heli*g*l_cm^2/(J_eq_p+m_heli*l_cm^2);

% Matrix F3 of F(x)
F3 = zeros(n,n^3);
F3(3, 16) = -m_heli*l_cm^2/(J_eq_p+m_heli*l_cm^2);
F3(4, 12) = 2*m_heli*l_cm^2/(J_eq_y+m_heli*l_cm^2);

% Matrix F4 of F(x)
F4 = zeros(n,n^4);
F4(3, 4^3) = -1/24*m_heli*g*l_cm/(J_eq_p+m_heli*l_cm^2);
F4(4, 4^3) = 1/2*(K_yp*u1o*m_heli*l_cm^2/(J_eq_y+m_heli*l_cm^2)^2 + ...
        K_yy*u2o*m_heli*l_cm^2/(J_eq_y+m_heli*l_cm^2)^2);

% Matrix G0 of G(x)
G0 = [                    0,                     0;
                      0,                     0;
      K_pp/(J_eq_p+m_heli*l_cm^2), K_py/(J_eq_p+m_heli*l_cm^2);
      K_yp/(J_eq_y+m_heli*l_cm^2), K_yy/(J_eq_y+m_heli*l_cm^2)];

% Matrix G1 of G(x)
G1 = zeros(n,m*n^1);
```

```
% Matrix G2 of G(x)
G2 = zeros(n,m*n^2);

% Matrix G3 of G(x)
G3 = zeros(n,m*n^3);

% Matrix G4 of G(x)
G4 = zeros(n,m*n^4);

% Matrix H1 of H(x)
H1 = eye(n);

% Matrix H2 of H(x)
H2 = zeros(n, n^2);

% Matrix H3 of H(x)
H3 = zeros(n, n^3);

% Matrix H4 of H(x)
H4 = zeros(n, n^4);

%% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ Simulation/Experiment Parameters
%%
theta_0deg = -40.5; % Initial Pitch Angle (deg)
theta_0 = theta_0deg*pi/180;

% Initial conditions
x10 = theta_0;    x20 = 0;    x30 = 0;    x40 = 0;
x0 = [x10;
      x20;
      x30;
      x40];

u10  = 0;    u20  = 0;
u0 = [u10;
      u20];

Tf = 220;  % Stop Time
Te = 1e-2; % Sampling Period

%% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ Computing
Pi's %%
Max_p = 4; % Maximum order of truncation generating Pp's
```

MATLAB code of calculation of control gain matrices for 2-DOF helicopter-model set-up

```
% Order of truncation 1: Term P1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
p = 1;

[P, L, G] = care(F1, G0, H1'*Q*H1, R);
P1 = chol(P);

% Scalar alpha %%%%%%%%%%%%%%%%%%%%%%%%%%%%
alpha = 0.1;

% Order of truncation 2: Term P2
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
p = 2;

IFp2 = kron((F1-G0/R*G0'*P)', eye(n^2));
% Term of Hp21 (i.e. 1st term of Hp2)
% i = 1 or j = 1
D1  = jDiffMatrix(n, 1);

% (i,j) = (1,1)
P11 = PijMatrix(P, P1, alpha, 1, 1);
V11 = Vec2Mat(vec(P11'*P11*D1), n, n)';
% (i,j,b,c) = (1,1,1,1) and (k,d) = (0,1)
W110 = Vec2Mat(vec(V11'*G0), n, m)';
% (i,j,b,c) = (1,1,1,1) and (k,d) = (1,0)
W111 = Vec2Mat(vec(V11'*G1), n^2, m)';
Hp21 = vec(W110'/R*W111) + vec(W111'/R*W110);

% Term of Hp22 (i.e. 2nd term of Hp2)
% (i,j) = (1,1) and k = 2
Hp22 = vec(V11'*F2) + vec(F2'*V11);

% Term of Hp23 (i.e. 3rd term of Hp2)
% (i,j) = (1,2) and (i,j) = (2,1)
Hp23 = vec(H1'*Q*H2) + vec(H2'*Q*H1);

% Total Term H2
Hp2 = Hp21 - Hp22 - Hp23;

% D_{p+1}^(n)
D3 = jDiffMatrix(n, 3);

T2 = NonRed2RedMat_of4thOrder(2); % At order p=2
T2p  = (T2'*T2)\T2';
TT2p = kron(T2p, eye(n))*D3;
```

```
vP2t = 1/(2*alpha)*((TT2p*TT2p')\TT2p)/IFp2*Hp2;

alpha_2 = size(T2);
P2t = Vec2Mat(vP2t, n, alpha_2(2));
P2 = P2t*T2p;

% Order of truncation 3: Term P3
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
p = 3;

IFp3 = kron((F1-G0/R*G0'*P)', eye(n^3));

Hp31 = zeros(n^(p+1),1);
Hp32 = zeros(n^(p+1),1);
Hp33 = zeros(n^(p+1),1);

% Term of Hp31 (i.e. 1st term of Hp3)
for i=1:p-1,%p-1=2
   for j=1:p-1,%p-1=2
      switch i
         case 1,
            Pi = P1;
         case 2,
            Pi = P2;
         otherwise
            disp('index out of range!');
      end
      switch j
         case 1,
            Pj = P1;
         case 2,
            Pj = P2;
         otherwise
            disp('index out of range!');
      end
      Pij = PijMatrix(P, Pi, alpha, i, j);
      Pji = PijMatrix(P, Pj, alpha, j, i);
      Dj  = jDiffMatrix(n, j);
      Vij = Vec2Mat(vec(Pij'*Pji*Dj), n^(i+j-1), n)';
      for k=0:p-1,%p-1=2
         switch k
            case 0,
               Gk = G0;
            case 1,
               Gk = G1;
            case 2,
```

```
            Gk = G2;
         otherwise
            disp('index out of range!');
      end
      Wijk = Vec2Mat(vec(Vij'*Gk), n^(i+j+k-1), m)';
      for b=1:p-1,%p-1=2
         for c=1:p-1,%p-1=2
            switch b
               case 1,
                  Pb = P1;
               case 2,
                  Pb = P2;
               otherwise
                  disp('index out of range!');
            end
            switch c
               case 1,
                  Pc = P1;
               case 2,
                  Pc = P2;
               otherwise
                  disp('index out of range!');
            end
            Pbc = PijMatrix(P, Pb, alpha, b, c);
            Pcb = PijMatrix(P, Pc, alpha, c, b);
            Dc = jDiffMatrix(n, c);
            Vbc = Vec2Mat(vec(Pbc'*Pcb*Dc), n^(b+c-1), n)';
            for d=0:p-1,%p-1=2
               switch d
                  case 0,
                     Gd = G0;
                  case 1,
                     Gd = G1;
                  case 2,
                     Gd = G2;
                  otherwise
                     disp('index out of range!');
               end
               Wbcd = Vec2Mat(vec(Vbc'*Gd), n^(b+c+d-1), m)';
               if (i+j+k+b+c+d==p+3),
                  Hp31 = Hp31 + vec(Wijk'/R*Wbcd);
               end
            end
         end
      end
   end
end
```

```
end

% Term of Hp32 (i.e. 2nd term of Hp3)
for i=1:p-1,%p-1=2
    for j=1:p-1,%p-1=2
        switch i
            case 1,
                Pi = P1;
            case 2,
                Pi = P2;
            otherwise
                disp('index out of range!');
        end
        switch j
            case 1,
                Pj = P1;
            case 2,
                Pj = P2;
            otherwise
                disp('index out of range!');
        end
        Pij = PijMatrix(P, Pi, alpha, i, j);
        Pji = PijMatrix(P, Pj, alpha, j, i);
        Dj  = jDiffMatrix(n, j);
        Vij = Vec2Mat(vec(Pij'*Pji*Dj), n^(i+j-1), n)';
        for k=1:p,%p=3
            switch k
                case 1,
                    Fk = F1;
                case 2,
                    Fk = F2;
                case 3,
                    Fk = F3;
                otherwise
                    disp('index out of range!');
            end
            if (i+j+k==p+2),
                Hp32 = Hp32 + vec(Vij'*Fk) + vec(Fk'*Vij);
            end
        end
    end
end

% Term of Hp33 (i.e. 3rd term of Hp3)
for i=1:p,
    switch i
        case 1,
```

```
         Hi = H1;
      case 2,
         Hi = H2;
      case 3,
         Hi = H3;
      otherwise
         disp('index out of range!');
   end
   for j=1:p,
      switch j
         case 1,
            Hj = H1;
         case 2,
            Hj = H2;
         case 3,
            Hj = H3;
         otherwise
            disp('index out of range!');
      end
      if (i+j==p+1),
         Hp33 = Hp33 + vec(Hi'*Q*Hj);
      end
   end
end

% Total Term H3
Hp3 = Hp31 - Hp32 - Hp33;

% D_{p+1}^(n)
D4 = jDiffMatrix(n, p+1);

T3 = NonRed2RedMat_of4thOrder(3); % At order p=3
T3p  = (T3'*T3)\T3';
TT3p = kron(T3p, eye(n))*D4;
vP3t = 1/(2*alpha)*((TT3p*TT3p')\TT3p)/IFp3*Hp3;

alpha_3 = size(T3);
P3t = Vec2Mat(vP3t, n, alpha_3(2));
P3 = P3t*T3p;

% Order of truncation 4: Term P4
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
p = 4;

IFp4 = kron((F1-G0/R*G0'*P)', eye(n^4));
```

```
Hp41 = zeros(n^(p+1),1);
Hp42 = zeros(n^(p+1),1);
Hp43 = zeros(n^(p+1),1);

% Term of Hp41 (i.e. 1st term of Hp4)
for i=1:p-1,%p-1=3
  for j=1:p-1,%p-1=3
    switch i
      case 1,
        Pi = P1;
      case 2,
        Pi = P2;
      case 3,
        Pi = P3;
      otherwise
        disp('index out of range!');
    end
    switch j
      case 1,
        Pj = P1;
      case 2,
        Pj = P2;
      case 3,
        Pj = P3;
      otherwise
        disp('index out of range!');
    end
    Pij = PijMatrix(P, Pi, alpha, i, j);
    Pji = PijMatrix(P, Pj, alpha, j, i);
    Dj  = jDiffMatrix(n, j);
    Vij = Vec2Mat(vec(Pij'*Pji*Dj), n^(i+j-1), n)';
    for k=0:p-1,%p-1=3
      switch k
        case 0,
          Gk = G0;
        case 1,
          Gk = G1;
        case 2,
          Gk = G2;
        case 3,
          Gk = G3;
        otherwise
          disp('index out of range!');
      end
      Wijk = Vec2Mat(vec(Vij'*Gk), n^(i+j+k-1), m)';
      for b=1:p-1,%p-1=3
        for c=1:p-1,%p-1=3
```

```
          switch b
             case 1,
                Pb = P1;
             case 2,
                Pb = P2;
             case 3,
                Pb = P3;
             otherwise
                disp('index out of range!');
          end
          switch c
             case 1,
                Pc = P1;
             case 2,
                Pc = P2;
             case 3,
                Pc = P3;
             otherwise
                disp('index out of range!');
          end
          Pbc = PijMatrix(P, Pb, alpha, b, c);
          Pcb = PijMatrix(P, Pc, alpha, c, b);
          Dc = jDiffMatrix(n, c);
          Vbc = Vec2Mat(vec(Pbc'*Pcb*Dc), n^(b+c-1), n)';
          for d=0:p-1,%p-1=3
             switch d
                case 0,
                   Gd = G0;
                case 1,
                   Gd = G1;
                case 2,
                   Gd = G2;
                case 3,
                   Gd = G3;
                otherwise
                   disp('index out of range!');
             end
             Wbcd = Vec2Mat(vec(Vbc'*Gd), n^(b+c+d-1), m)';
             if (i+j+k+b+c+d==p+3),
                Hp41 = Hp41 + vec(Wijk'/R*Wbcd);
             end
          end
       end
    end
  end
 end
end
```

```
% Term of Hp42 (i.e. 2nd term of Hp4)
for i=1:p-1,%p-1=3
    for j=1:p-1,%p-1=3
        switch i
            case 1,
                Pi = P1;
            case 2,
                Pi = P2;
            case 3,
                Pi = P3;
            otherwise
                disp('index out of range!');
        end
        switch j
            case 1,
                Pj = P1;
            case 2,
                Pj = P2;
            case 3,
                Pj = P3;
            otherwise
                disp('index out of range!');
        end
        Pij = PijMatrix(P, Pi, alpha, i, j);
        Pji = PijMatrix(P, Pj, alpha, j, i);
        Dj  = jDiffMatrix(n, j);
        Vij = Vec2Mat(vec(Pij'*Pji*Dj), n^(i+j-1), n)';
        for k=1:p,%p=4
            switch k
                case 1,
                    Fk = F1;
                case 2,
                    Fk = F2;
                case 3,
                    Fk = F3;
                case 4,
                    Fk = F4;
                otherwise
                    disp('index out of range!');
            end
            if (i+j+k==p+2),
                Hp42 = Hp42 + vec(Vij'*Fk) + vec(Fk'*Vij);
            end
        end
    end
end
```

```
% Term of Hp43 (i.e. 3rd term of Hp4)
for i=1:p,
   switch i
      case 1,
         Hi = H1;
      case 2,
         Hi = H2;
      case 3,
         Hi = H3;
      case 4,
         Hi = H4;
      otherwise
         disp('index out of range!');
   end
   for j=1:p,
      switch j
         case 1,
            Hj = H1;
         case 2,
            Hj = H2;
         case 3,
            Hj = H3;
         case 4,
            Hj = H4;
         otherwise
            disp('index out of range!');
      end
      if (i+j==p+1),
         Hp43 = Hp43 + vec(Hi'*Q*Hj);
      end
   end
end

% Total Term H4
Hp4 = Hp41 - Hp42 - Hp43;

% D_{p+1}^(n)
D5 = jDiffMatrix(n, p+1);

T4 = NonRed2RedMat_of4thOrder(4); % At order p=4
T4p  = (T4'*T4)\T4';
TT4p = kron(T4p, eye(n))*D5;
vP4t = 1/(2*alpha)*((TT4p*TT4p')\TT4p)/IFp4*Hp4;

alpha_4 = size(T4);
```

MATLAB code of calculation of control gain matrices for 2-DOF helicopter-model set-up

```
P4t = Vec2Mat(vP4t, n, alpha_4(2));
P4 = P4t*T4p;

%% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ Control
Gains %%

% Term Kp's with p^bar = 1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%
pbar = 1;% Using only P (Order of truncation 1 for Pp)
% p from 1 to pbar = 1

p = 1;
% Then i = j = 1 and k = 0
Kp1_pbar1 = R\W110;

% Term Kp's with p^bar = 2
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%
pbar = 2;% Using P to order of truncation 2 for Pp)
% p from 1 to pbar = 2

p = 1;
% Case of i = j = 1 and k = 0
Kp1_pbar2 = R\W110;

p = 2;
% Case of i = 2, j = 1 and k = 0
P21 = PijMatrix(P, P2, alpha, 2, 1);
P12 = PijMatrix(P, P1, alpha, 1, 2);
D2 = jDiffMatrix(n, 2);
V21 = Vec2Mat(vec(P21'*P12*D1), n^2, n)';
W210 = Vec2Mat(vec(V21'*G0), n^2, m)';
% Case of i = 1, j = 2 and k = 0
V12 = Vec2Mat(vec(P12'*P21*D2), n^2, n)';
W120 = Vec2Mat(vec(V12'*G0), n^2, m)';

W2 = W210 + W120;
Kp2_pbar2 = R\W2;

% Term Kp's with p^bar = 3
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%
pbar = 3;% Using P to order of truncation 3 for Pp)
% p from 1 to pbar = 3
```

```
p = 1;
% Case of i = j = 1 and k = 0
Kp1_pbar3 = R\W110;

p = 2;
% Case of i = 2, j = 1 and k = 0 and case of i = 1, j = 2 and k = 0

Kp2_pbar3 = R\W2;

p = 3;
% Case of i = 3, j = 1 and k = 0
P31 = PijMatrix(P, P3, alpha, 3, 1);
P13 = PijMatrix(P, P1, alpha, 1, 3);
V31 = Vec2Mat(vec(P31'*P13*D1), n^3, n)';
W310 = Vec2Mat(vec(V31'*G0), n^3, m)';
% Case of i = 2, j = 2 and k = 0
P22 = PijMatrix(P, P2, alpha, 2, 2);
V22 = Vec2Mat(vec(P22'*P22*D2), n^3, n)';
W220 = Vec2Mat(vec(V22'*G0), n^3, m)';
% Case of i = 1, j = 1 and k = 0
V13 = Vec2Mat(vec(P13'*P31*D3), n^3, n)';
W130 = Vec2Mat(vec(V13'*G0), n^3, m)';

W3 = W310 + W220 + W130;
Kp3_pbar3 = R\W3;

% Term Kp's with p^bar = 4
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%
pbar = 4;% Using P to order of truncation 4 for Pp)
% p from 1 to pbar = 4

p = 1;
% Case of i = j = 1 and k = 0
Kp1_pbar4 = R\W110;

p = 2;
% Case of i = 2, j = 1 and k = 0 and case of i = 1, j = 2 and k = 0
Kp2_pbar4 = R\W2;

p = 3;
% Case of i = 3, j = 1 and k = 0, case of i = 2, j = 2 and k = 0, and
% case of i = 1, j = 1 and k = 0
Kp3_pbar4 = R\W3;
```

```
p = 4;
% Case of i = 4, j = 1 and k = 0,
P41 = PijMatrix(P, P4, alpha, 4, 1);
P14 = PijMatrix(P, P1, alpha, 1, 4);
V41 = Vec2Mat(vec(P41'*P14*D1), n^4, n)';
W410 = Vec2Mat(vec(V41'*G0), n^4, m)';
% Case of i = 3, j = 2 and k = 0,
P32 = PijMatrix(P, P3, alpha, 3, 2);
P23 = PijMatrix(P, P2, alpha, 2, 3);
V32 = Vec2Mat(vec(P32'*P23*D2), n^4, n)';
W320 = Vec2Mat(vec(V32'*G0), n^4, m)';
% Case of i = 2, j = 3 and k = 0, and
V23 = Vec2Mat(vec(P23'*P32*D3), n^4, n)';
W230 = Vec2Mat(vec(V23'*G0), n^4, m)';
% Case of i = 1, j = 4 and k = 0
V14 = Vec2Mat(vec(P14'*P41*D4), n^4, n)';
W140 = Vec2Mat(vec(V14'*G0), n^4, m)';

W4 = W410 + W320 + W230 + W140;
Kp4_pbar4 = R\W4;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Choose Order of truncation p = 1, 2, 3 or 4.~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
p = 4;%Kepp it equal to 4

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
switch p
   case 1,
      % Apply the following
      % Term Kp's with p =
1.%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%

      % We obtain, with only p = 1:
      p = 1;% Using only P (Order of truncation 1 for Pp)
      Kp_1 = Kp1_pbar1;
      Kp_2 = zeros(m, n^2);
      Kp_3 = zeros(m, n^3);
      Kp_4 = zeros(m, n^4);
      Kp_5 = zeros(m, n^5);
      Kp_6 = zeros(m, n^6);
      Kp_7 = zeros(m, n^7);
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

   case 2,
      % Apply the following
      % Term Kp's with p =
2.%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%

      % We obtain, with only p = 2:
      p = 2;% Using only P (Order of truncation 2 for Pp)
      Kp_1 = Kp1_pbar2;
      Kp_2 = Kp2_pbar2;
      Kp_3 = zeros(m, n^3);
      Kp_4 = zeros(m, n^4);
      Kp_5 = zeros(m, n^5);
      Kp_6 = zeros(m, n^6);
      Kp_7 = zeros(m, n^7);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

   case 3,
      % Apply the following
      % Term Kp's with p = 3.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

      % We obtain, with only p = 3:
      p = 3;% Using only P (Order of truncation 3 for Pp)

      Kp_1 = Kp1_pbar3;
      Kp_2 = Kp2_pbar3;
      Kp_3 = Kp3_pbar3;
      Kp_4 = zeros(m, n^4);
      Kp_5 = zeros(m, n^5);
      Kp_6 = zeros(m, n^6);
      Kp_7 = zeros(m, n^7);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

   case 4,
      % Apply the following
      % Term Kp's with p = 4.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
        % We obtain, with only p = 4:
        p = 4;% Using only P (Order of truncation 4 for Pp)

        Kp_1 = Kp1_pbar4;
        Kp_2 = Kp2_pbar4;
        Kp_3 = Kp3_pbar4;
        Kp_4 = Kp4_pbar4;
        Kp_5 = zeros(m, n^5);
        Kp_6 = zeros(m, n^6);
        Kp_7 = zeros(m, n^7);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    otherwise
        disp('index out of range!');
end

%% Run for Order of K with p's (from 1 to p).~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
for Order_of_K = 1:p,


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%
    sim('Heli2dofNLOCSimulation')
    disp('Simulation done..')


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%
    load States.mat states;
    load Cmd.mat cmd;
    load Jopt.mat Jopt;

    switch Order_of_K
        case 1,
            X_lin = states;
            U_lin = cmd;
            J_lin = Jopt;

            save X_lin.mat X_lin;
            save U_lin.mat U_lin;
            save J_lin.mat J_lin
        case 2,
```

```
        X_KP2 = states;
        U_KP2 = cmd;
        J_KP2 = Jopt;

        save X_KP2.mat X_KP2;
        save U_KP2.mat U_KP2;
        save J_KP2.mat J_KP2;
      case 3,
        X_KP3 = states;
        U_KP3 = cmd;
        J_KP3 = Jopt;

        save X_KP3.mat X_KP3;
        save U_KP3.mat U_KP3;
        save J_KP3.mat J_KP3;
      case 4,
        X_KP4 = states;
        U_KP4 = cmd;
        J_KP4 = Jopt;

        save X_KP4.mat X_KP4;
        save U_KP4.mat U_KP4;
        save J_KP4.mat J_KP4;
      otherwise
        disp('index out of range!');
   end
end

save Pd.mat Pd
save Pdp.mat Pdp
save Yd.mat Yd

%% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
End .. %%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

F Simulation results for other desired trajectories of 2-DOF helicopter-model set-up

F.1 Constant desired pitch angle of -30 degree

In the following we present the simulation results for a desired yaw angle of 0 degree, desired pitch angle of -30 degree and an initial condition of the pitch angle of -40.5 degrees for four controllers (Linear, 2^{nd}, 3^{rd} and 4^{th} orders).

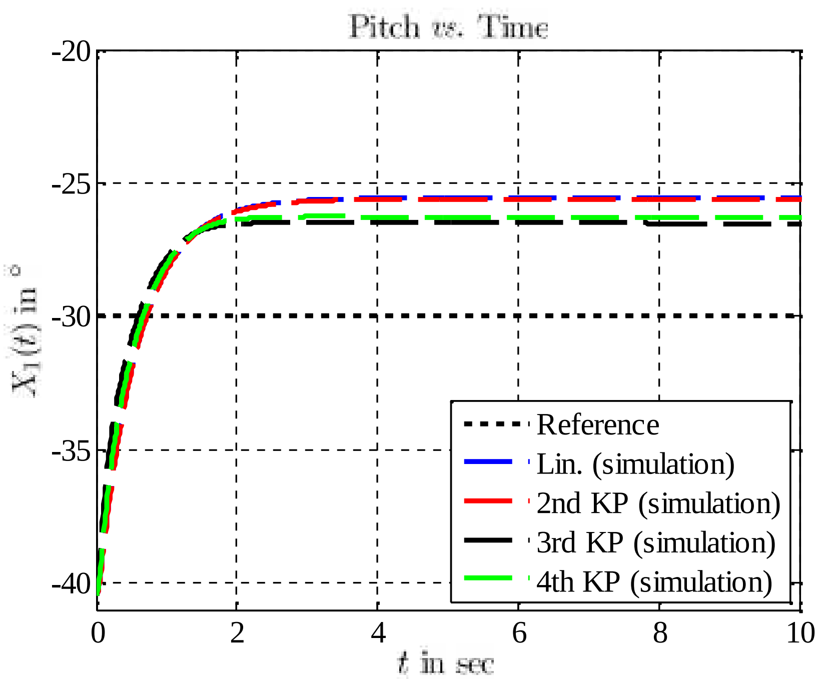

Figure F1. Pitch evolution *vs.* time for desired pitch angle of -30 degree

The simulations show in Figure 7.7 that the four controllers stabilize the system around the pitch angle of -26 degree. The same simulations show also that the four controllers have the same behaviour.

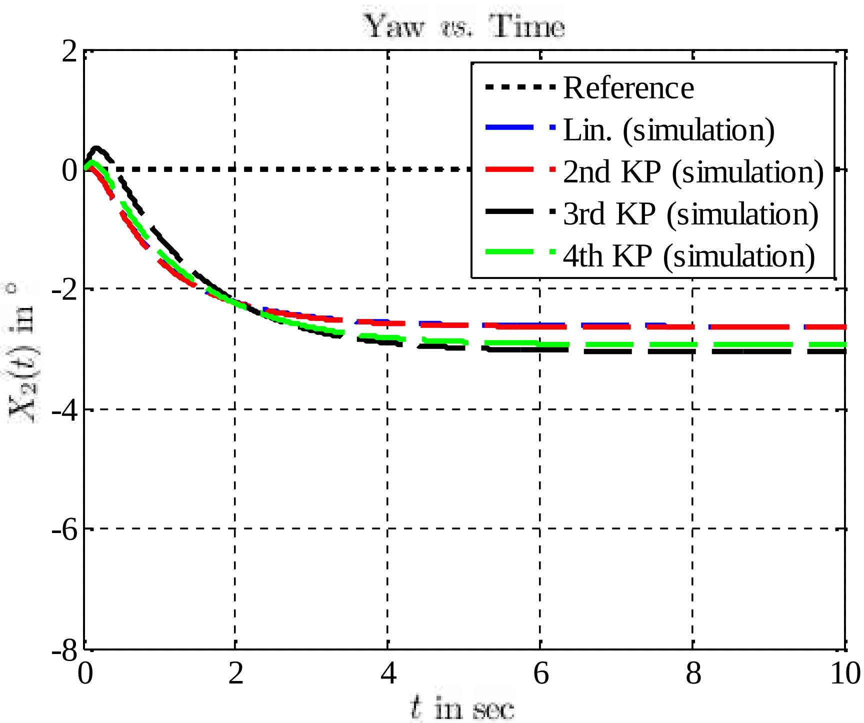

Figure.F2. Yaw evolution *vs.* time for desired pitch angle of -30 degree

Figure 7.8 shows that the four controllers stabilize the system around a yaw angle of -3 degree and they are similar.

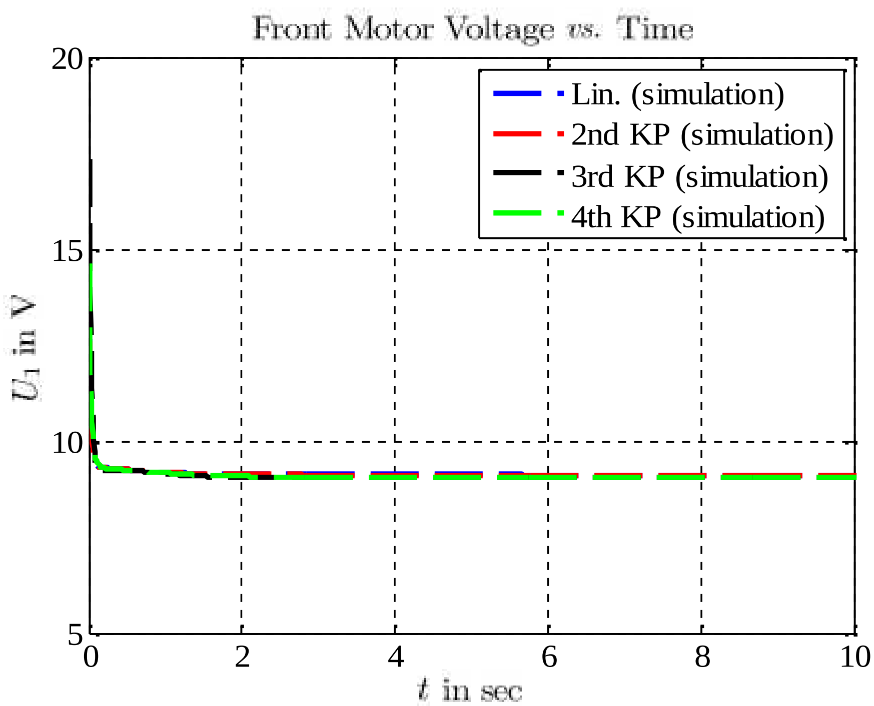

Figure F3. Front motor voltage evolution *vs*. time for desired pitch angle of -30 degree

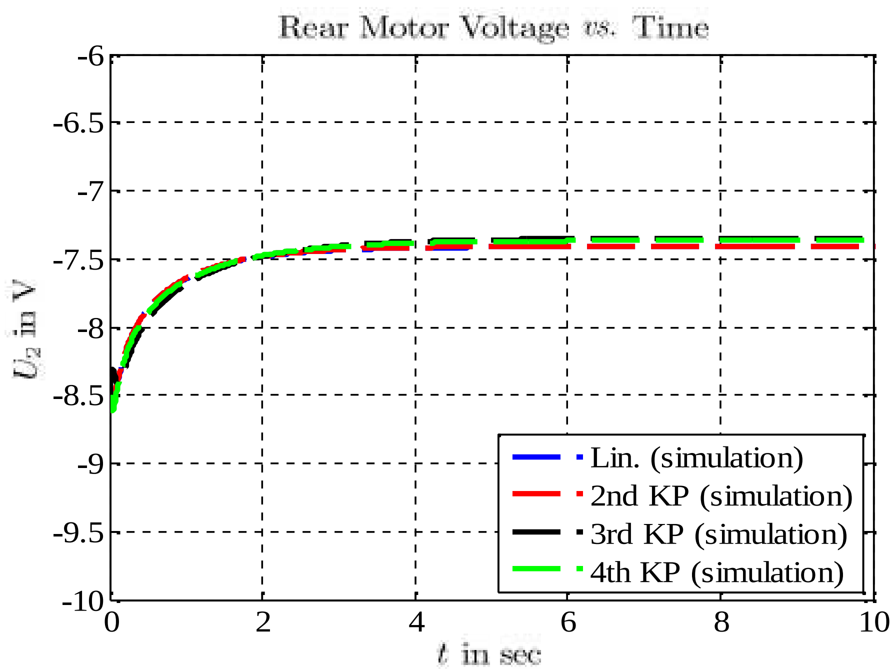

Figure.F4. Rear motor voltage evolution *vs.* time for desired pitch angle of -30 degree

Figures F3 and F4 show the input voltages of the front and rear motors. They present almost the same behaviour for the four controllers.

F.2 Square signal desired pitch angle of 0.05 Hz frequency and 10 degree amplitude

In the following, we present the simulation results for a desired yaw angle of 0 degree, desired pitch angle of a square signal of frequency 0.05 Hz and amplitude of 10 degree, with an initial condition of the pitch angle of -40.5 degrees for four controllers: Linear, 2^{nd} , 3^{rd} and 4^{th} truncation orders.

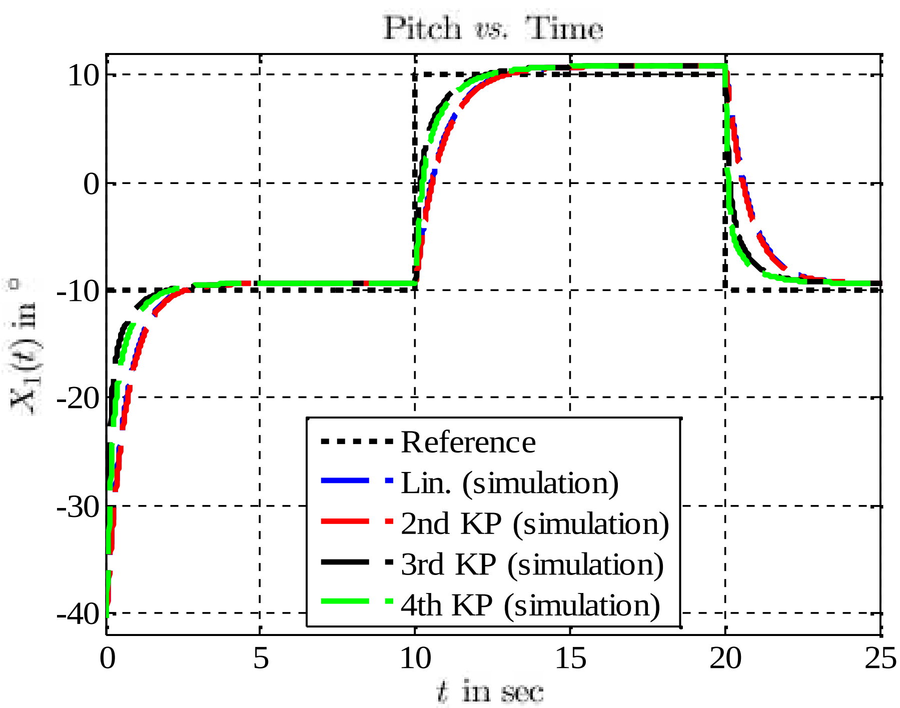

Figure F5. Pitch evolution *vs.* time for desired pitch angle of square signal of frequency 0.05 Hz and amplitude of 10 degree

The simulation results for the pitch angle show that the four controllers stabilize the helicopter around the desired signal with an advantage for 3^{rd} and 4^{th} order ones presenting a closer behaviour to the reference and better settling time.

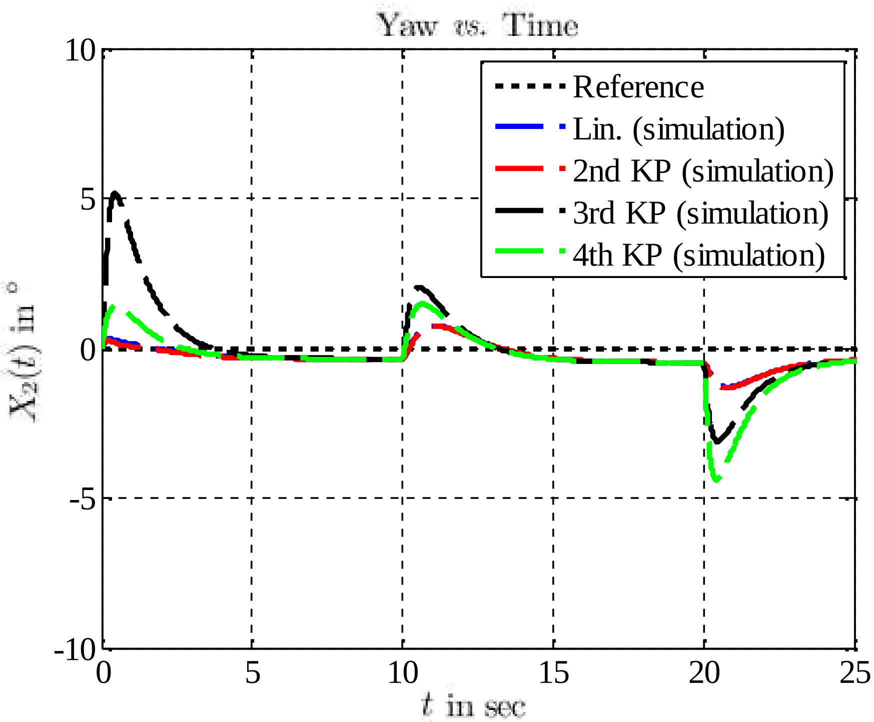

Figure F6. Yaw evolution *vs.* time for desired pitch angle of square signal of frequency 0.05 Hz and amplitude of 10 degree

The simulation results of the yaw angle show that the four controllers stabilize the helicopter around the desired yaw angle of zero degree, while the 3^{rd} and 4^{th} order ones present un important overshoot to reach the equilibrium, the linear and 2^{nd} order controllers present a less steady state error.

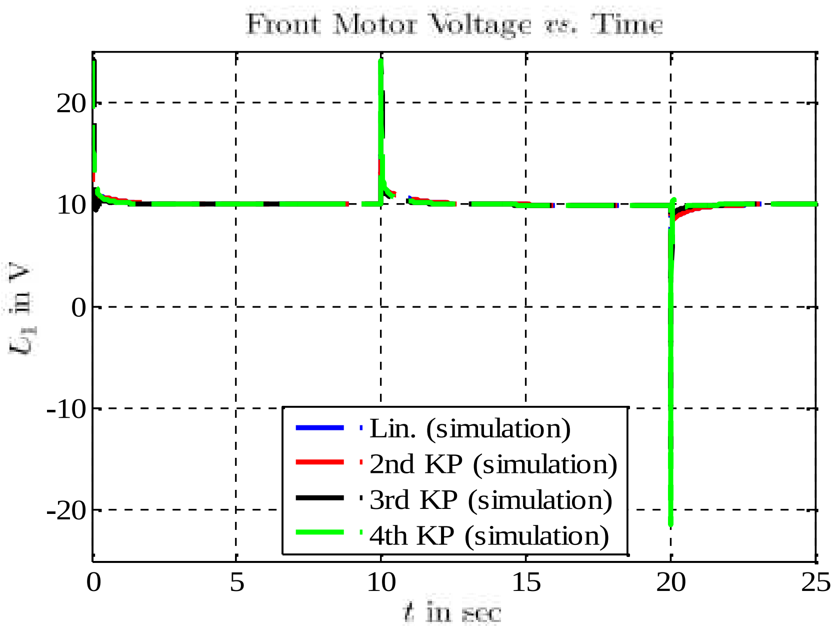

Figure F7. Front motor voltage evolution *vs.* time for desired pitch angle of square signal of frequency 0.05 Hz and amplitude of 10 degree

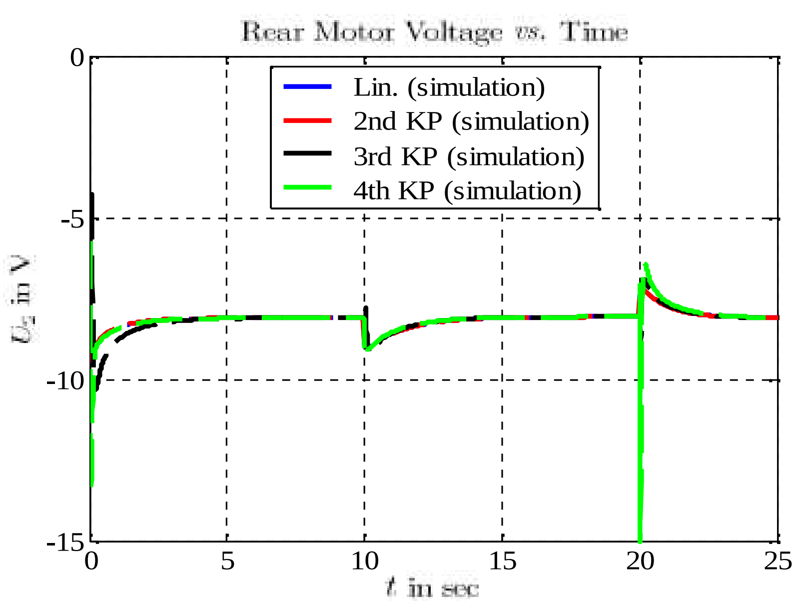

Figure F8. Rear Motor Voltage evolution *vs.* time for desired pitch angle of square signal of frequency 0.05 Hz and amplitude of 10 degree

The simulation results of the front and rear motor voltages show that the 3^{rd} and 4^{th} order controllers require a higher voltage, then more energy to stabilize the helicopter around the desired pitch and yaw angles.

F.3 Square signal desired pitch angle of 0.02 Hz frequency and 20 degree amplitude

In the following, we present the simulation results for a desired yaw angle of 0 degree, desired pitch angle of a square signal of frequency 0.02 Hz and amplitude of 20 degree and for an initial condition of the pitch angle of -40.5 degrees for four controllers: Linear, 2^{nd} , 3^{rd} and 4^{th} truncation order.

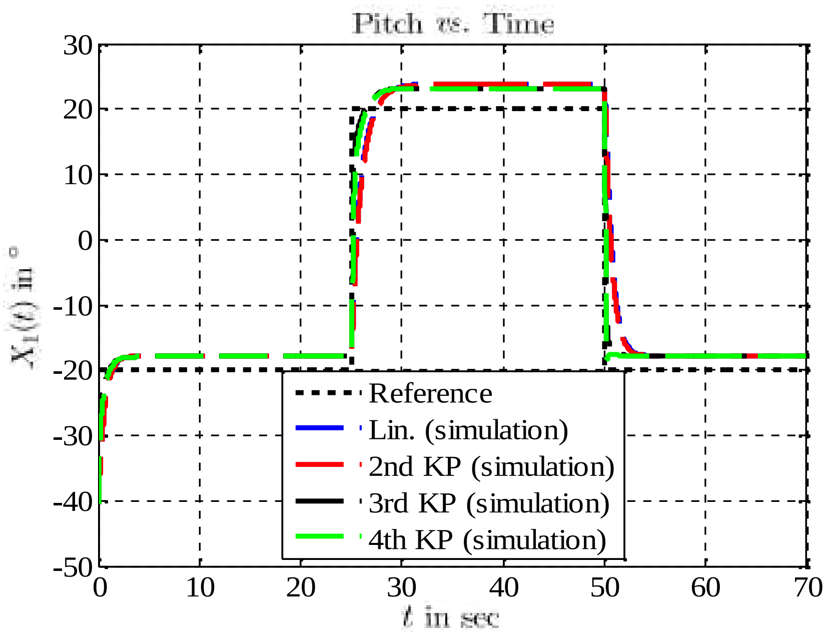

Figure F9. Pitch evolution *vs.* time for desired pitch angle of square signal of frequency 0.02 Hz and amplitude of 20 degree

The simulation results of the pitch angle show that the four controllers stabilize the helicopter around the desired signal with the advantage for the 3^{rd} and 4^{th} ones presenting a closer behaviour to the reference and better performance in terms of settling time.

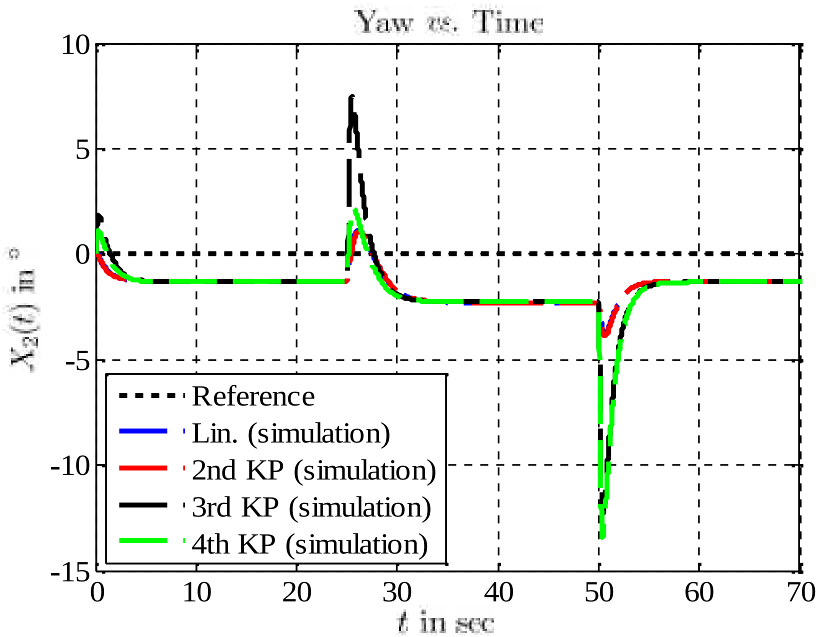

Figure F10. Yaw evolution *vs.* time for desired pitch angle of square signal of frequency 0.02 Hz and amplitude of 20 degree

The simulation results of the yaw angle show that the four controllers stabilize the helicopter around the desired yaw angle of zero degree. The first and second order controllers present a closer behaviour to the reference, while 3^{rd} and 4^{th} ones present a higher overshoot.

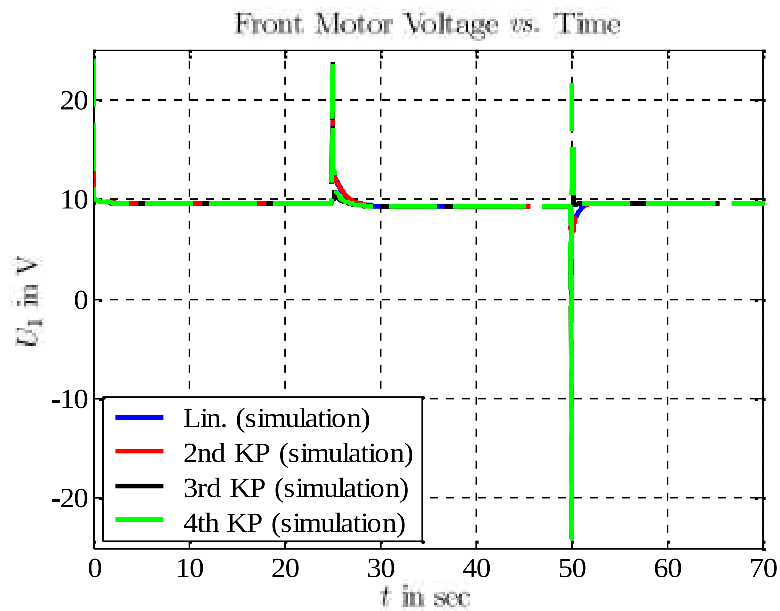

Figure F11. Front motor voltage evolution *vs.* time for desired pitch angle of square signal of frequency 0.02 Hz and amplitude of 20 degree

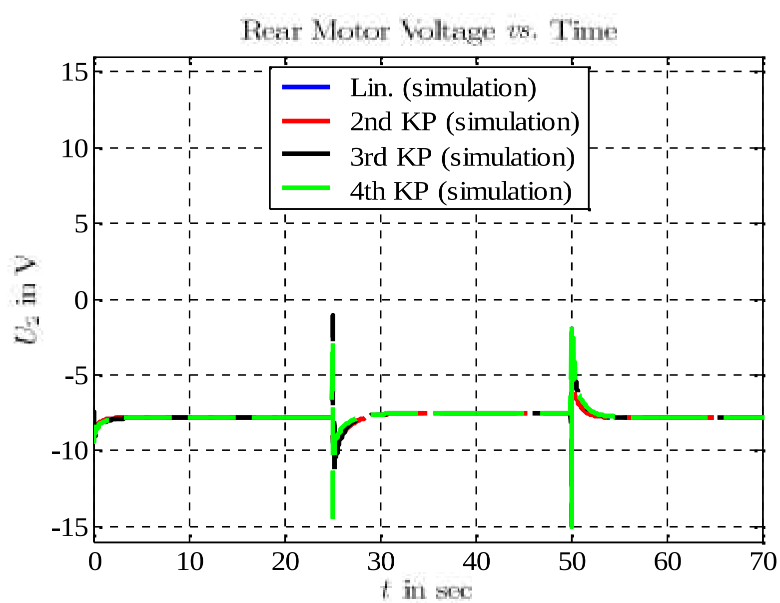

Figure F12. Rear motor voltage evolution *vs.* time for desired pitch angle of square signal of frequency 0.02 Hz and amplitude of 20 degree

The simulation results for the input controls, the front and rear motor voltages show that 3^{rd} and 4^{th} order controllers require a higher voltage, then more energy to stabilize the helicopter around the desired pitch and yaw angles.

F.4 Sine signal desired pitch angle of 0.05 Hz frequency and 10 degree amplitude

In the following, we present the simulation results for a desired yaw angle of 0 degree, desired pitch angle of a sine signal of frequency 0.05 Hz and amplitude of 10 degree with an initial condition of the pitch angle of -40.5 degrees for the four controllers: linear, 2^{nd} , 3^{rd} and 4^{th} truncation order.

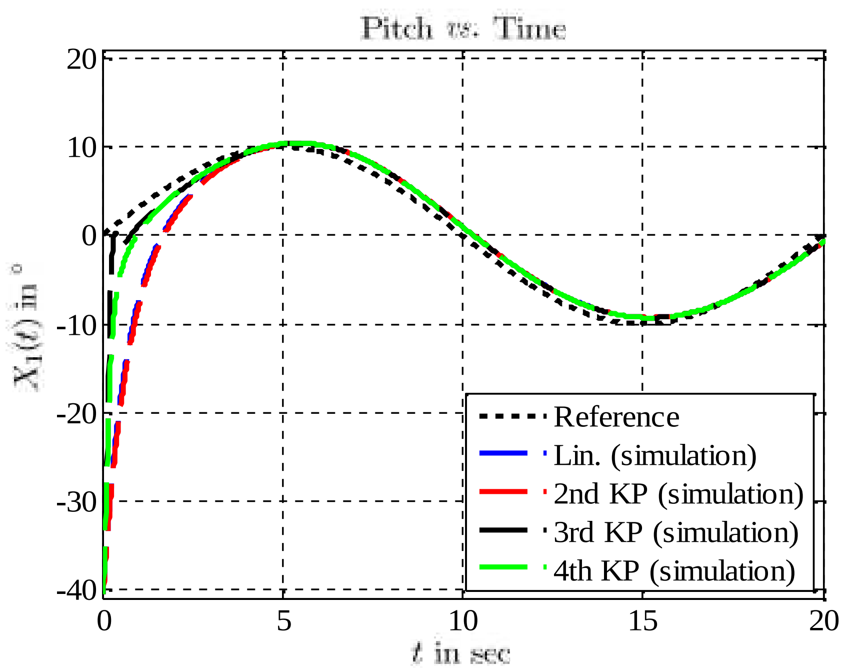

Figure F13. Pitch evolution *vs.* time for desired pitch angle of sine signal of frequency 0.05 Hz and amplitude of 10 degree

The simulation results of the pitch angle show that the four controllers stabilize the helicopter around the desired signal with the advantage for 3^{rd} and 4^{th} order ones presenting a better performance in terms of rise time.

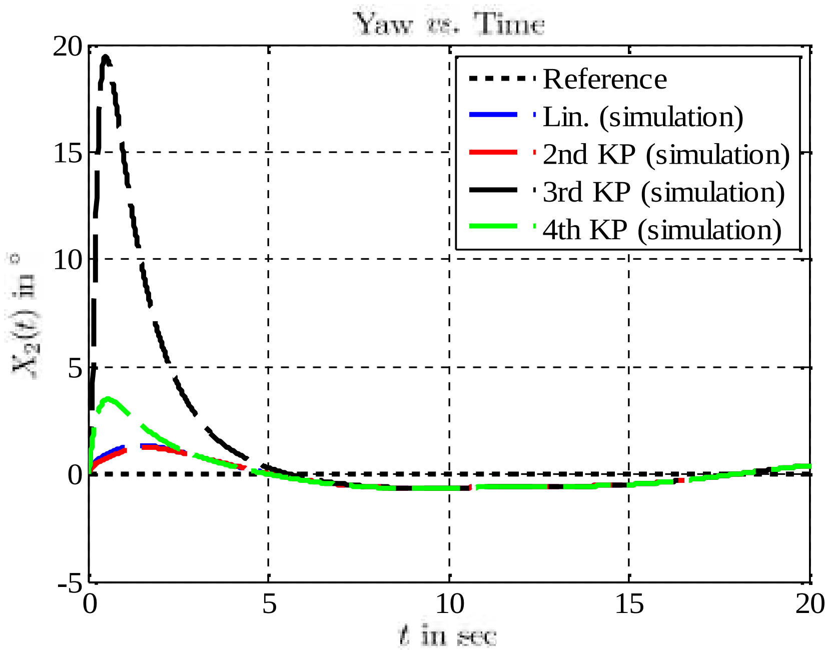

Figure F14 Yaw evolution *vs.* time for desired pitch angle of sine signal of frequency 0.05 Hz and amplitude of 10 degree

The simulation results of the yaw angle show that the four controllers stabilize the helicopter around the desired angle of zero degree with the advantage for the linear, 2^{nd} and 4^{th} order ones, presenting a better performance in terms of overshoot.

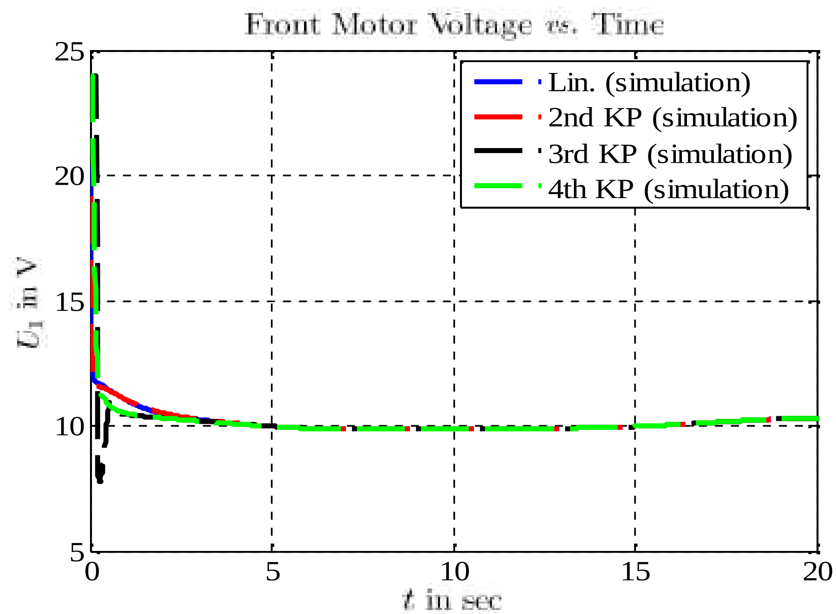

Figure F15 Front motor voltage evolution *vs.* time for desired pitch angle of sine signal of frequency 0.05 Hz and amplitude of 10 degree

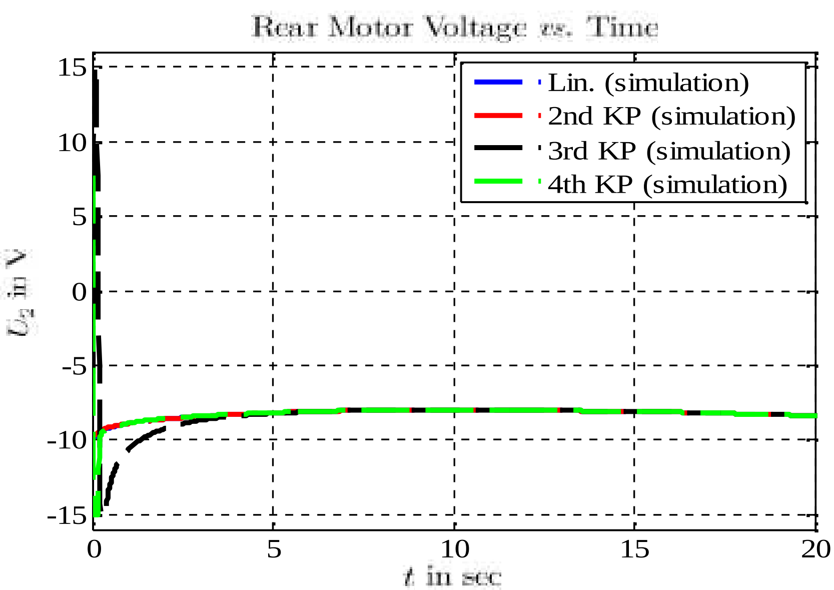

Figure F16 Rear motor voltage *vs.* time for desired pitch angle of sine signal of frequency 0.05 Hz and amplitude of 10 degree

The simulation results for the input controls, *i.e.*, the front and rear motor voltages show that 3^{rd} and 4^{th} order controllers require a higher voltage, then more energy to stabilize the helicopter around the desired pitch and yaw angles.

F.5 Sine signal desired pitch angle of 0.02 Hz frequency and 20 degree amplitude

In the following, we present the simulation results for a desired yaw angle of 0 degree, desired pitch angle of a sine signal of frequency 0.02 Hz and amplitude of 20 degree, with an initial condition of the pitch angle of -40.5 degrees for the four controllers: Linear, 2^{nd} , 3^{rd} and 4^{th} truncation order.

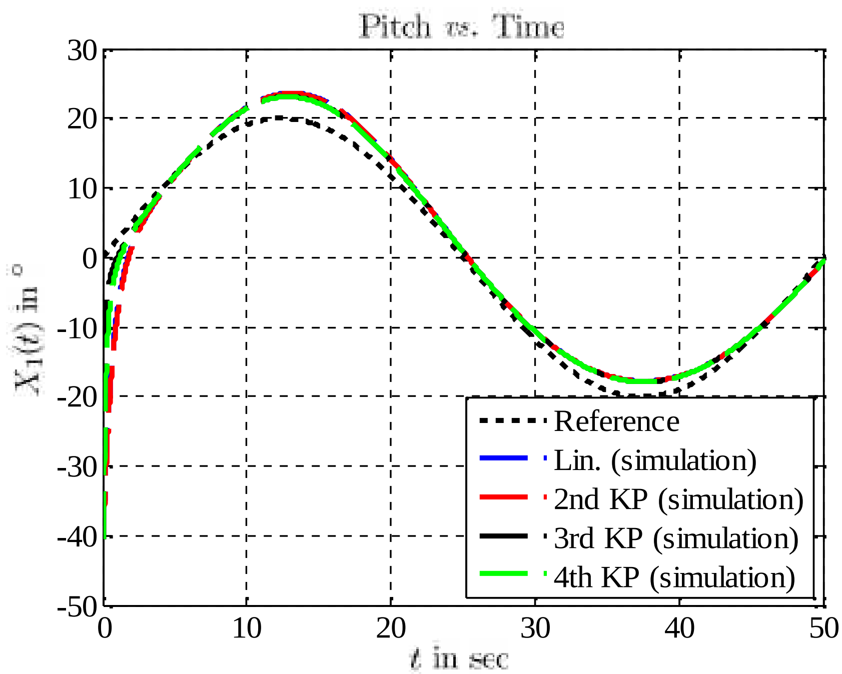

Figure F17 Pitch evolution *vs.* time for desired pitch angle of sine signal of frequency 0.02 Hz and amplitude of 20 degree

The simulation results of the pitch angle show that the four controllers stabilize the helicopter around the desired signal with the advantage for the 3^{rd} and the 4^{th} ones presenting a better performance in terms of rise time.

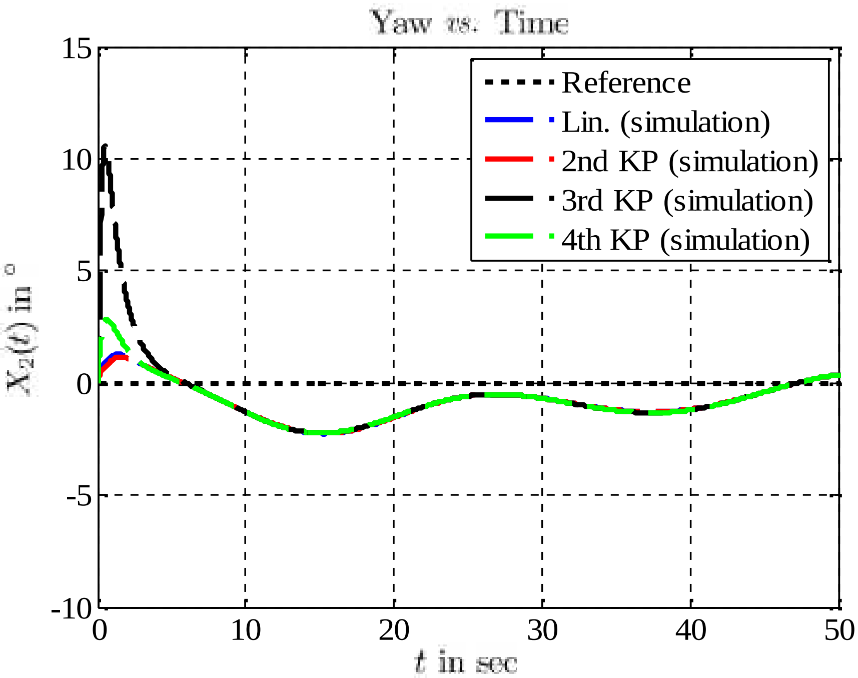

Figure F18 Yaw evolution *vs.* time for desired pitch angle of sine signal of frequency 0.02 Hz and amplitude of 20 degree

The simulation results of the yaw angle show that the four controllers stabilize the helicopter around the desired angle of zero degree with the advantage for the linear, 2^{nd} and 4^{th} order ones, presenting a better performance in terms of overshoot. We note that the proposed control design does not consider the transient dynamic behaviour (*e.g.*, overshoot). The optimal control gain calculus minimizes a combination of the energy of error and control effort.

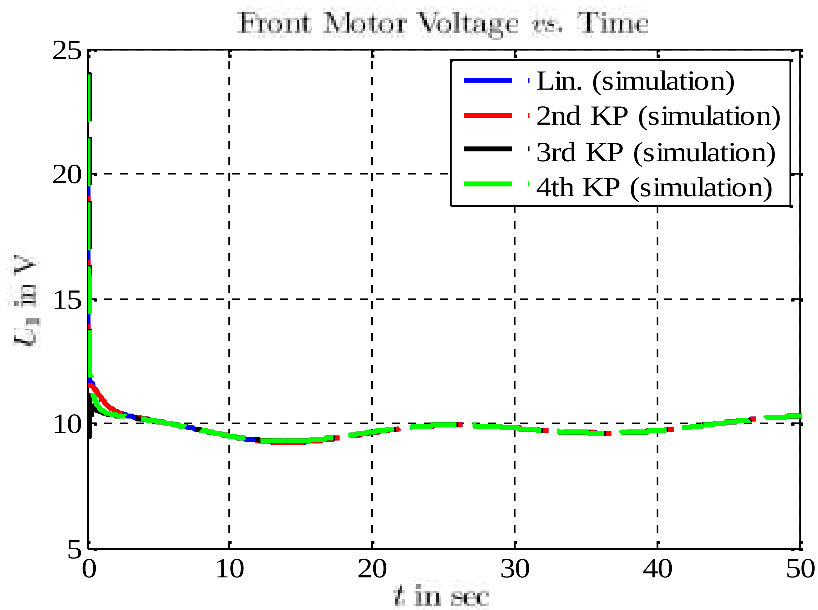

Figure F19 Front motor voltage evolution *vs.* time for desired pitch angle of sine signal of frequency 0.02 Hz and amplitude of 20 degree

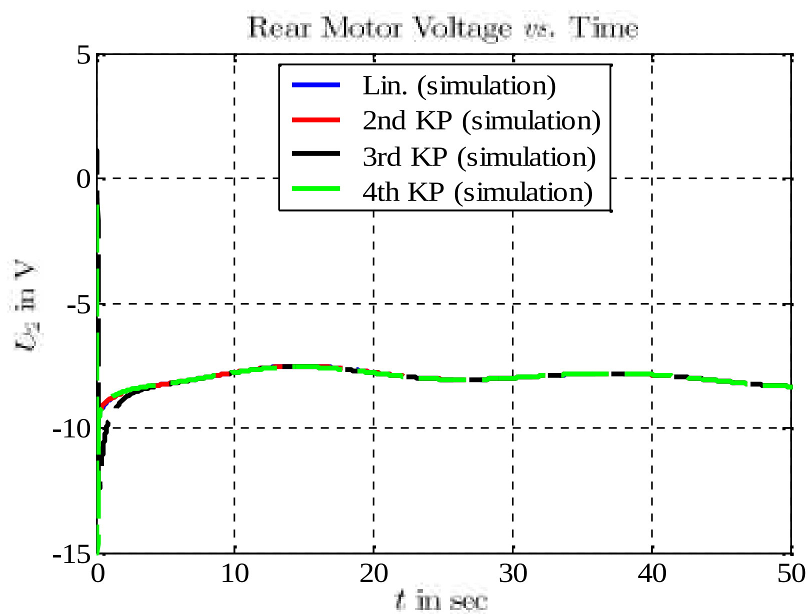

Figure F20 Rear motor voltage evolution *vs.* time for desired pitch angle of sine signal of frequency 0.02 Hz and amplitude of 20 degree

The simulation results for the input controls, *i.e.*, the front and rear motor voltages show that 3^{rd} and 4^{th} order controllers require a higher voltage, then more energy to stabilize the helicopter around the desired pitch and yaw angles.

F.6 Multi-step desired pitch angle

In the following, we present the simulation results for a desired yaw angle of 0 degree, desired pitch angle of a multi-step signal with an initial condition of the pitch angle of -40.5 degrees for the four controllers Linear, 2^{nd} , 3^{rd} and 4^{th} truncation orders.

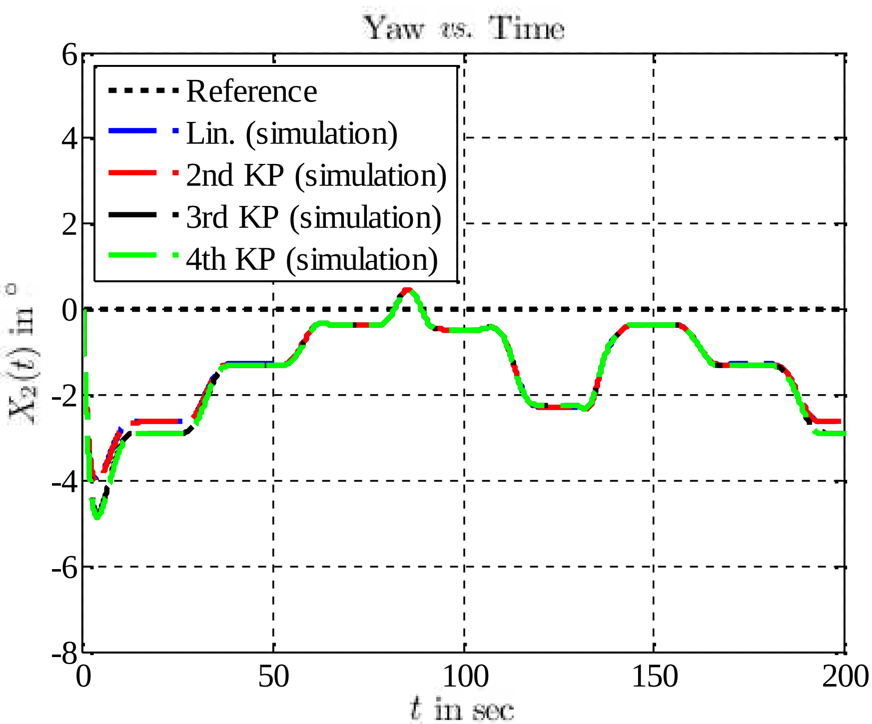

Figure F21 Yaw evolution *vs.* time for desired pitch angle of multi-steps signal

The simulation results of the yaw angle show that the four controllers behave similarly (*i.e.*, there is no major differences).

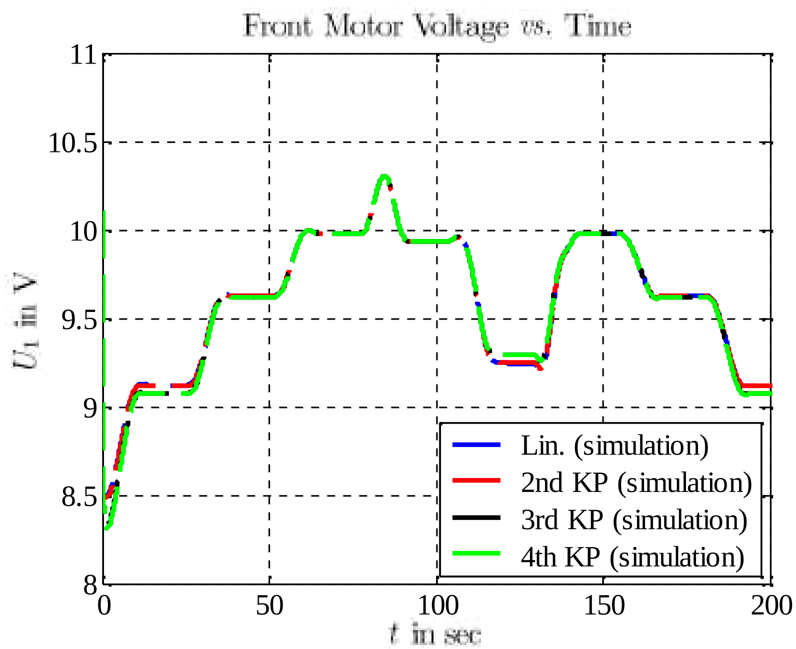

Figure F22 Front motor voltage evolution *vs.* time for desired pitch angle of multi-steps signal

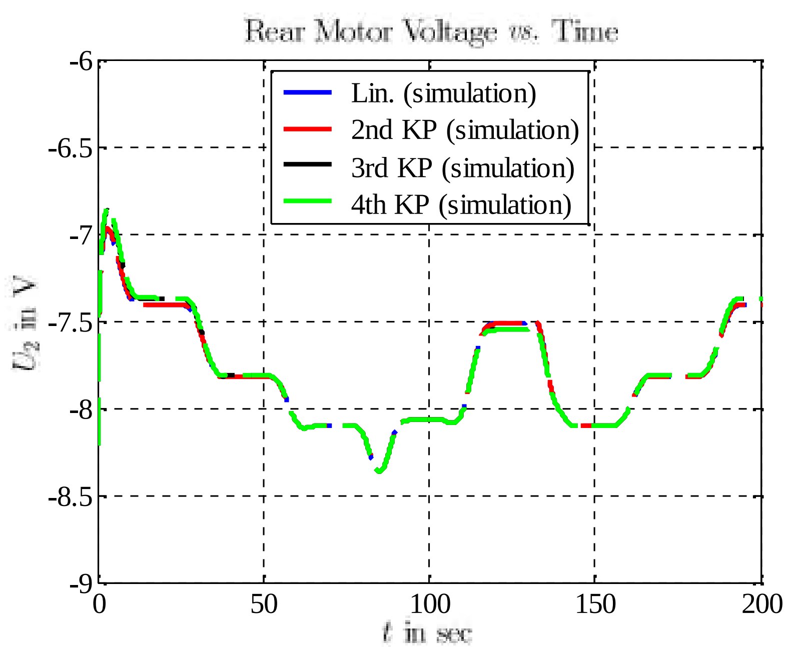

Figure F23 Rear motor voltage evolution *vs.* time for desired pitch angle of multi-steps signal

The simulation for the input controls, *i.e.*, the front and rear motor voltages show that for the four controllers, the input control signals have the same behaviour, and then, the same amount of energy is needed in order to stabilize the system around the desired trajectory.

In conclusion, the simulation of the behaviour of the designed four controllers for different trajectories show that as high as we go in the order of truncation, we need higher energy to stabilize the system but we see an improvement in the steady state error.

G Experimental results for other desired trajectories of the 2-DOF helicopter-model set-up

G.1 Constant desired pitch angle of -30 degree

In the following, we present the simulation results for a desired yaw angle of 0 degree, desired pitch angle of -30 degree with an initial condition of the pitch angle of -40.5 degrees for four controllers: Linear, 2^{nd} , 3^{rd} and 4^{th} truncation orders.

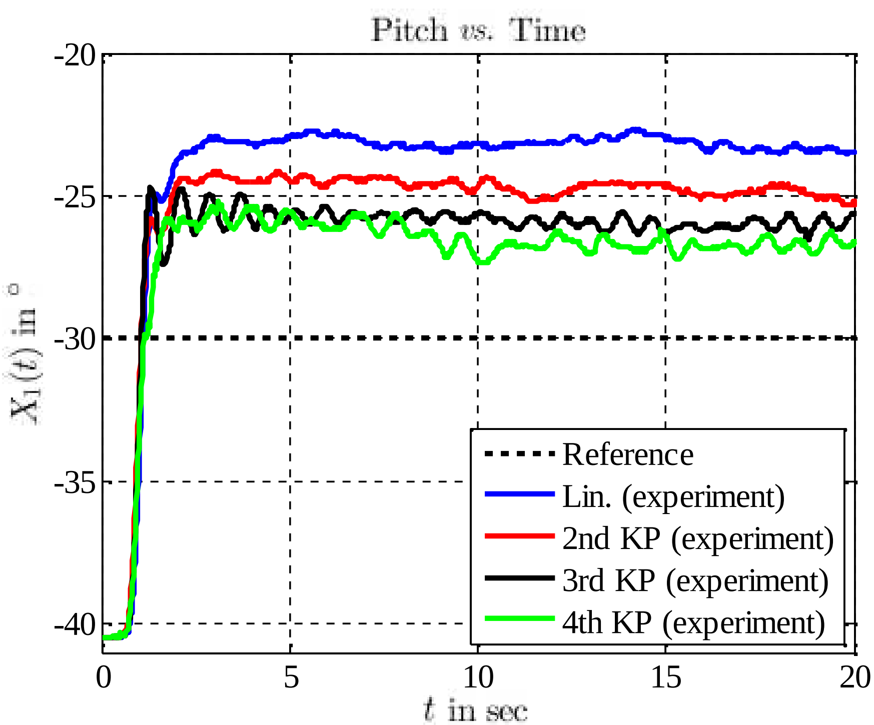

Figure G1 Pitch evolution *vs.* time for desired pitch angle of -30 degree

The experimental results show that the four controllers stabilize the 2-DOF helicopter set-up around a pitch angle of -25 degree. The depicted error due to the mathematical model approximation of the control law is reduced with truncation order. In comparison with simulation results, the experimental results present the same behaviour with closely the same error magnitudes. These results demonstrate the effectiveness of the high order KP-LF-based method.

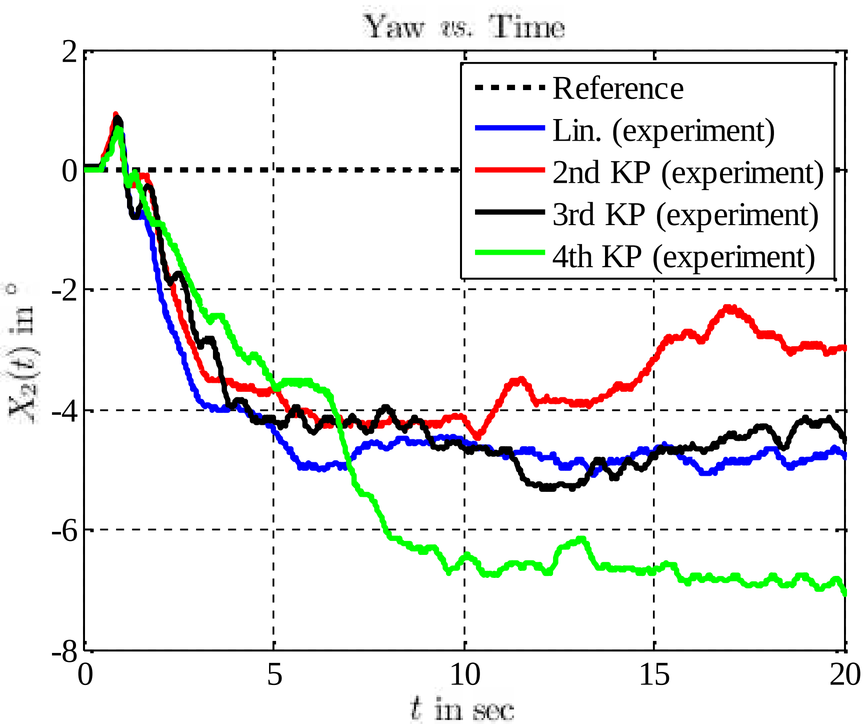

Figure G2 Yaw evolution *vs.* time for desired pitch angle of -30 degree

The experimental results show how the four controllers stabilize the 2-DOF helicopter set-up around a yaw angle within the range of -3 to -7 degrees. These errors are due to the mathematical model approximation of the control law. In comparison with simulation results, the experimental results present almost the same behaviour and closely same level of errors, with slight improvement with the 3rd order.

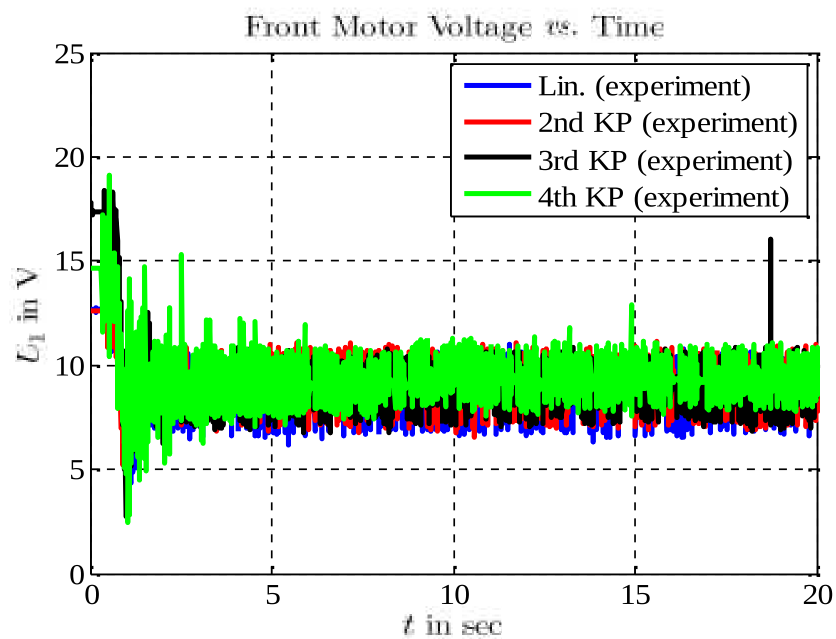

Figure G3 Front motor voltage evolution *vs.* time for desired pitch angle of -30 degree

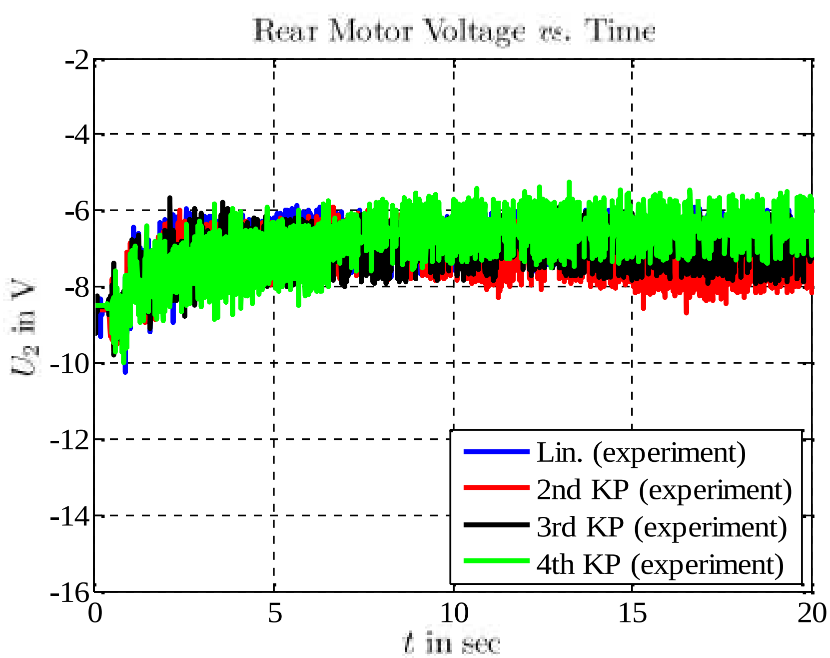

Figure G4 Rear motor voltage evolution *vs.* time for desired pitch angle of -30 degree

The experimental results show that for the input controls, *i.e.*, the front motor and rear motor voltages that the four controllers behave the same way: for the front motor the voltage fluctuate within the range of 7 to 12V and for the rear motor the voltage fluctuate within the range of -8 to -6V. We assume that these fluctuations are due to nonlinearities in the mathematical model of the system and some noise levels which are not controllable. In comparison with the simulation results, the experimental ones present the same general tendency with more fluctuations around the equilibrium input controls instead of a constant value.

G.2 Square signal desired pitch angle of 0.05 Hz frequency and 10 degree amplitude

In the following, we present the experimental results for a desired yaw angle of 0 degree, desired pitch angle of a square signal of frequency 0.05 Hz and amplitude of 10 degree with an initial condition of the pitch angle of -40.5 degrees for four controllers: Linear, 2^{nd}, 3^{rd} and 4^{th} truncation orders.

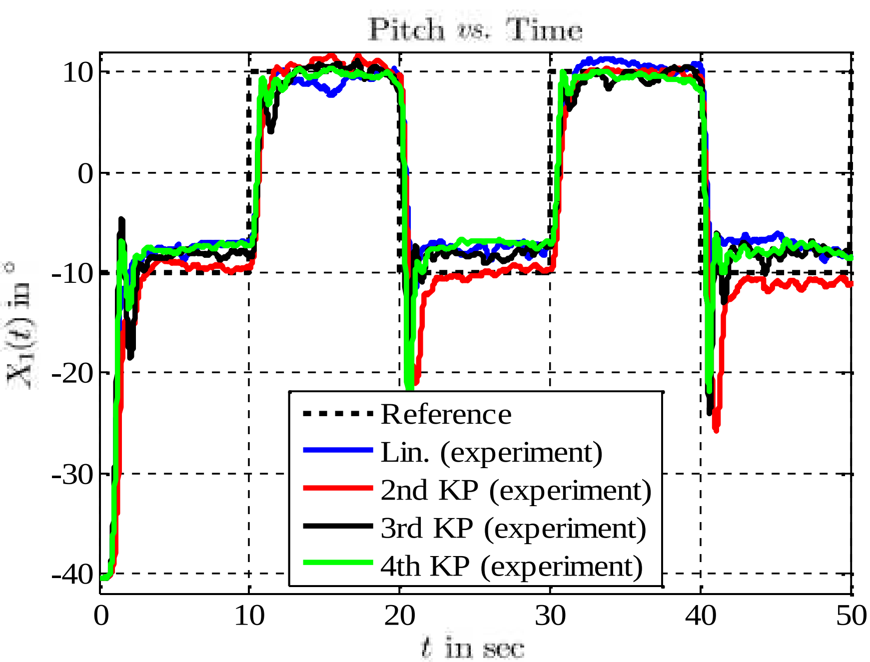

Figure G5 Pitch evolution *vs.* time for desired pitch angle of square signal of frequency 0.05 Hz and amplitude of 10 degree

The experimental results show that the four controllers behave the same way and stabilize the 2-DOF set-up around the desired square pitch angle signal with some overshoot when the signal is changing amplitude from 10 degrees to -10 degrees. The 2nd order controller represents the best performance in terms of accuracy.

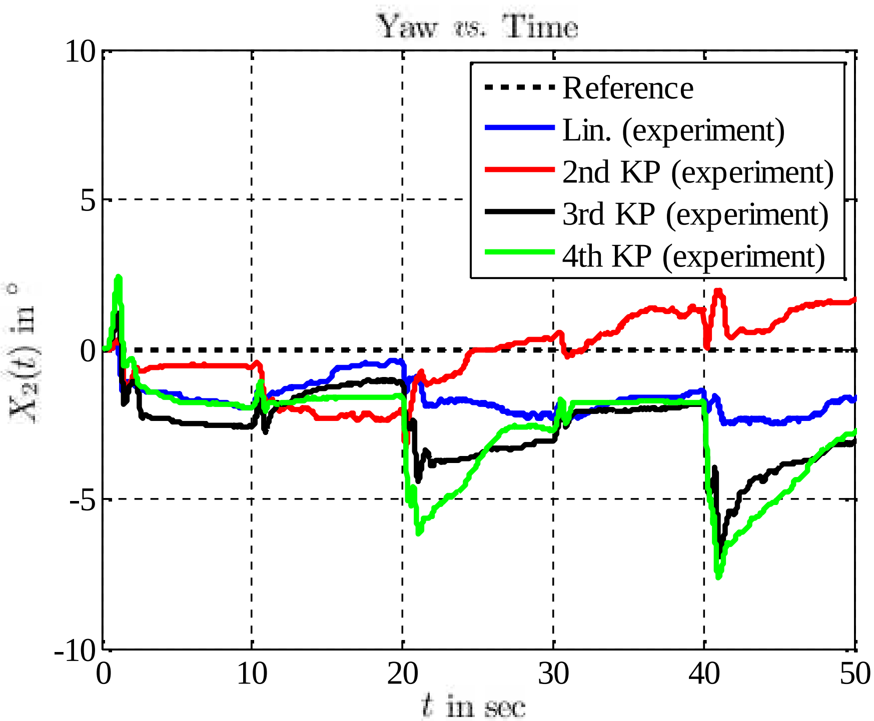

Figure G6 Yaw evolution *vs.* time for desired pitch angle of square signal of frequency 0.05 Hz and amplitude of 10 degree

The same experiments show that the four controllers stabilize the 2-DOF set-up around the desired yaw angle of 0 degree with an improvement of the second order one, presenting lower errors than the other controllers.

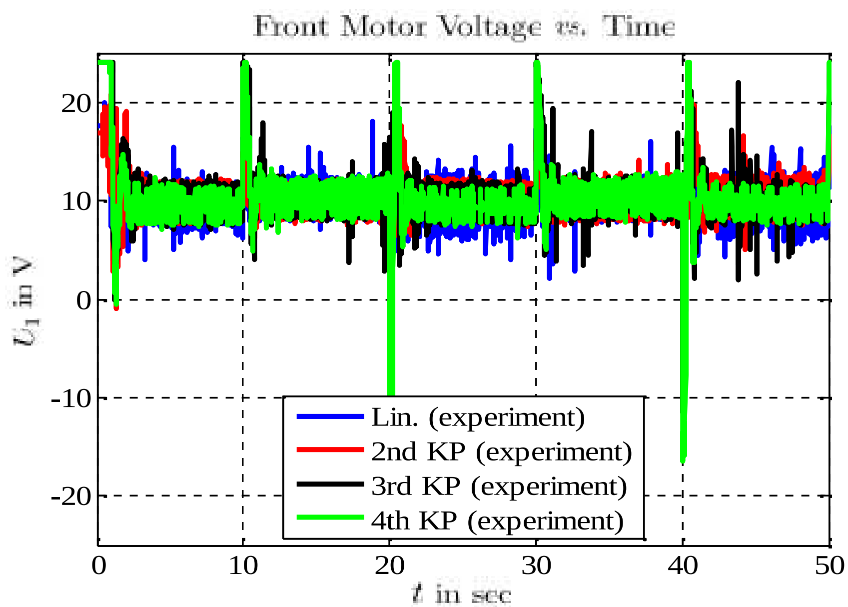

Figure G7 Front motor voltage evolution *vs.* time for desired pitch angle of square signal of frequency 0.05 Hz and amplitude of 10 degree

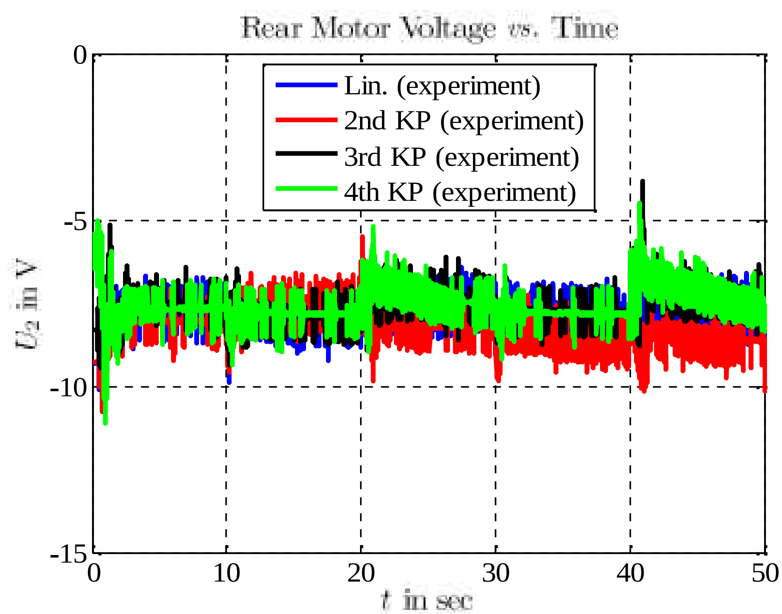

Figure G8 Rear motor voltage evolution *vs.* time for desired pitch angle of square signal of frequency 0.05 Hz and amplitude of 10 degree

In terms of input controls, the experiments show for the front motor voltage that the four controllers behave in the same way. But, the fourth order one presents a higher voltage variation when the signal is changing direction from 10 degrees to -10 degrees. For the rear motor voltage there is no major difference between the four controllers.

G.3 Square signal desired pitch angle of 0.02 Hz frequency and 20 degree amplitude

In the following, we present the experimental results for a desired yaw angle of 0 degree, desired pitch angle of a square signal of frequency 0.02 Hz and amplitude of 20 degree, with for an initial condition of the pitch angle of -40.5 degrees for four controllers: Linear, 2^{nd} , 3^{rd} and 4^{th} truncation orders.

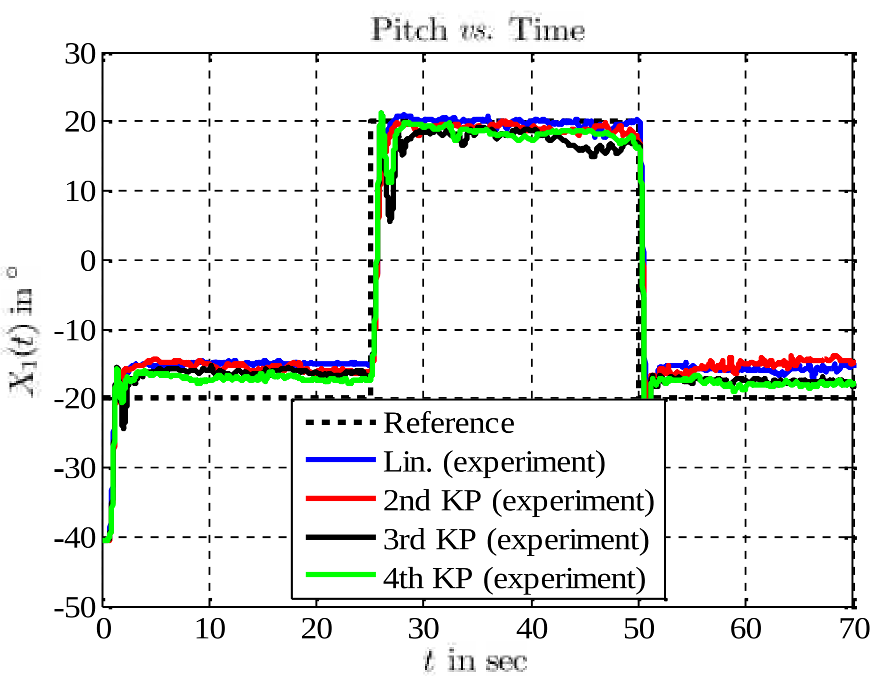

Figure G9 Pitch evolution *vs.* time for desired pitch angle of square signal of frequency 0.02 Hz and amplitude of 20 degree

The experimental result for the pitch angle show that the four controllers stabilize the 2-DOF set-up around the desired square signal with a slight improvement for the second order one, presenting a lower overshoot when the signal amplitude is changing from 20 degrees to -20 degrees.

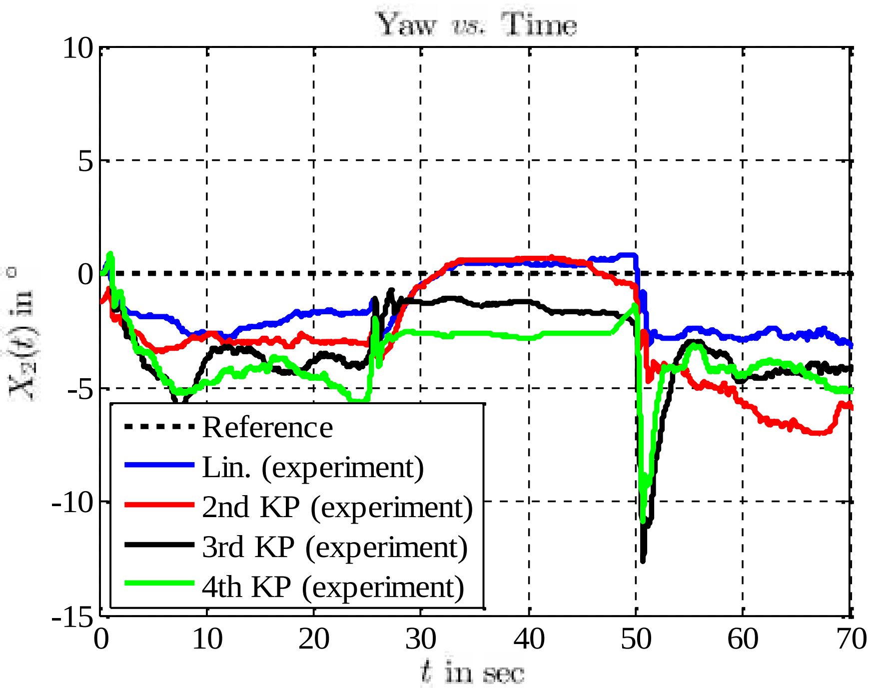

Figure G10 Yaw evolution *vs.* time for desired pitch angle of square signal of frequency 0.02 Hz and amplitude of 20 degree

The experimental results for the yaw angle show that the four controllers stabilize the system around the desired yaw angle of zero degree with errors of -5 to 0 degrees, with some overshoot for the third and fourth order controllers when changing the signal amplitude from 20 degrees to -20 degrees.

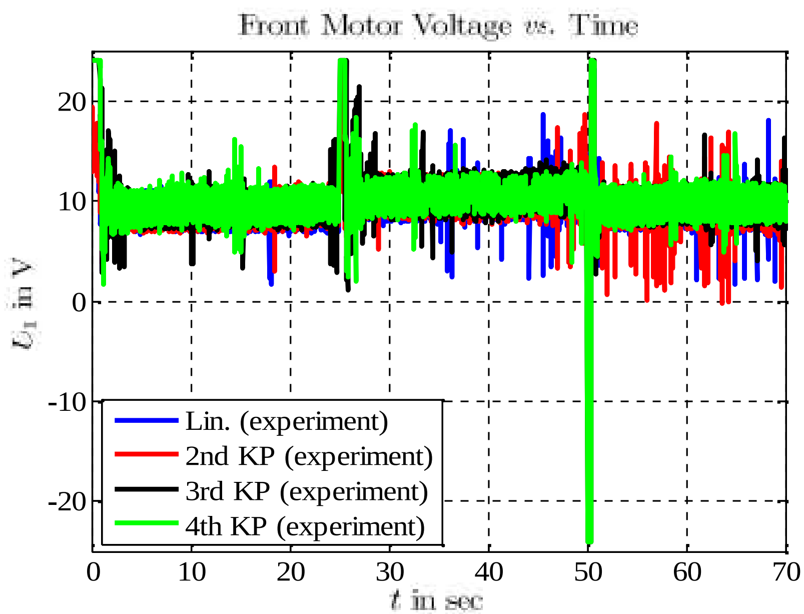

Figure G11 Front motor voltage evolution *vs.* time for desired pitch angle of square signal of frequency 0.02 Hz and amplitude of 20 degree

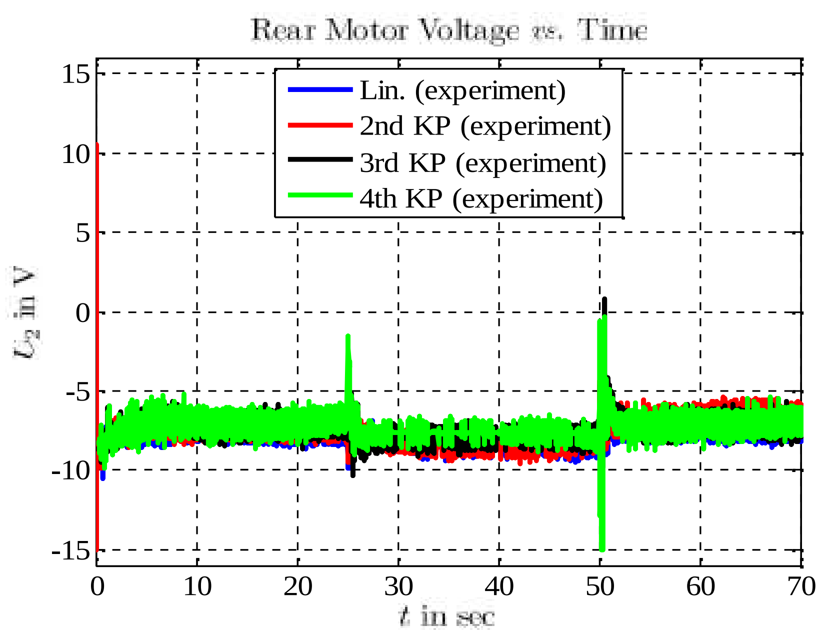

Figure G12 Rear motor voltage evolution *vs.* time for desired pitch angle of square signal of frequency 0.02 Hz and amplitude of 20 degree

The experimental results for the input controls, *i.e.*, front and rear motor voltages show that the four controllers behave almost in the same way and then consume almost the same amount of energy with the exception for the fourth order one which presents higher voltage for both motors when changing the desired pitch angle amplitude from 20 degrees to -20 degrees.

G.4 Sine signal desired pitch angle of 0.05 Hz frequency and 10 degree amplitude

In the following, we present the experimental results for a desired yaw angle of 0 degree, desired pitch angle of a sine signal of frequency 0.05 Hz and amplitude of 10 degree, with an initial condition of the pitch angle of -40.5 degrees for the four controllers: Linear, 2^{nd} , 3^{rd} and 4^{th} truncation orders.

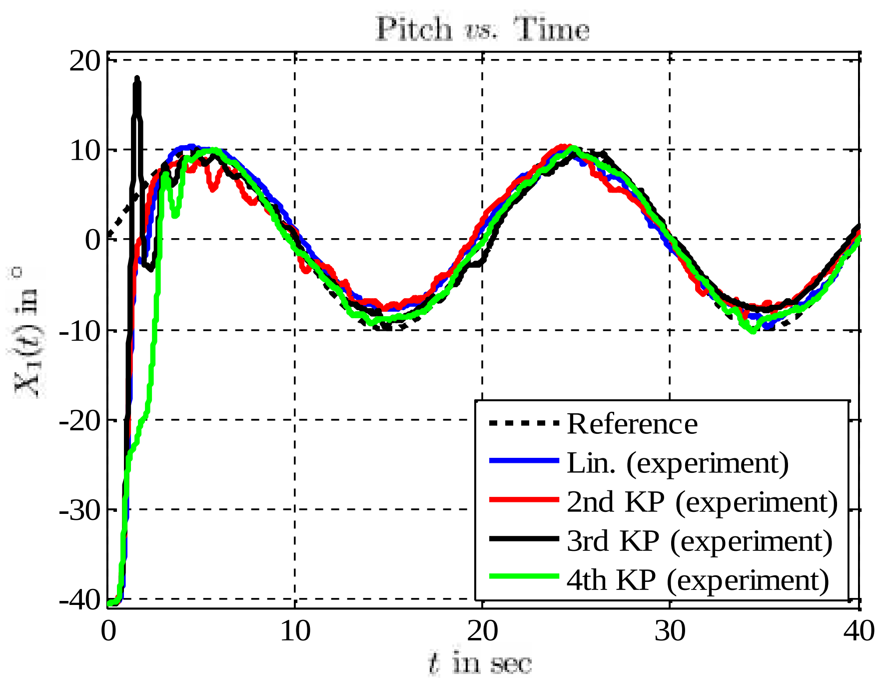

Figure G13 Pitch evolution *vs.* time for desired pitch angle of sine signal of frequency 0.05 Hz and amplitude of 10 degree

The experimental results for the pitch angle show that the four controllers stabilize the 2-DOF set-up around the desired sine pitch angle with a slight advantage for the second order one which presents a better performance than the other controllers in terms of overshoot, rise time and errors.

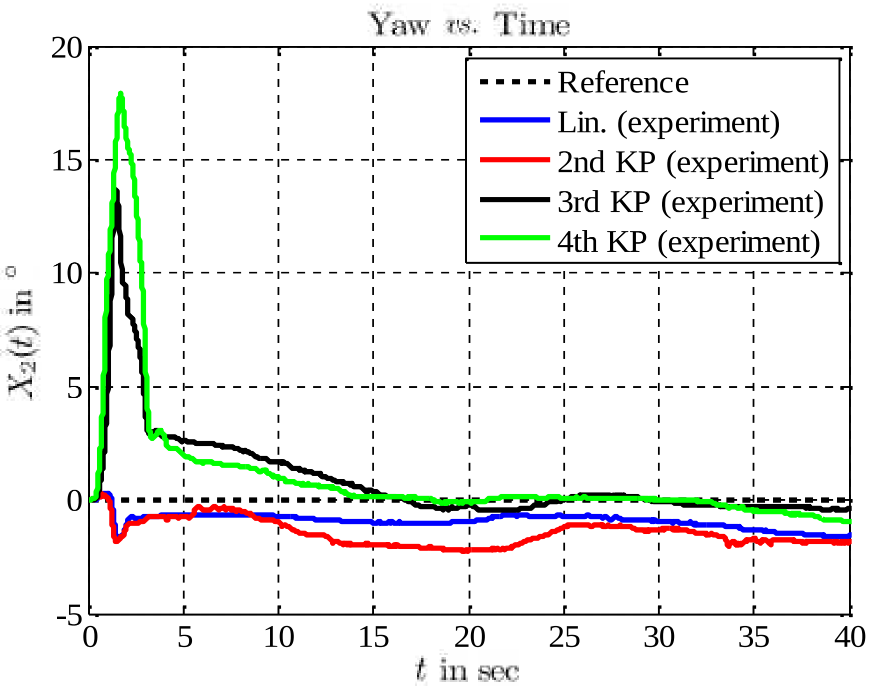

Figure G14 Yaw evolution *vs.* time for desired pitch angle of sine signal of frequency 0.05 Hz and amplitude of 10 degree

For the yaw angle, the simulation results show that the four controllers stabilize the 2-DOF set-up around the desired yaw angle of zero degree with a slight advantage for third and fourth order controllers in terms of errors despite an important overshoot.

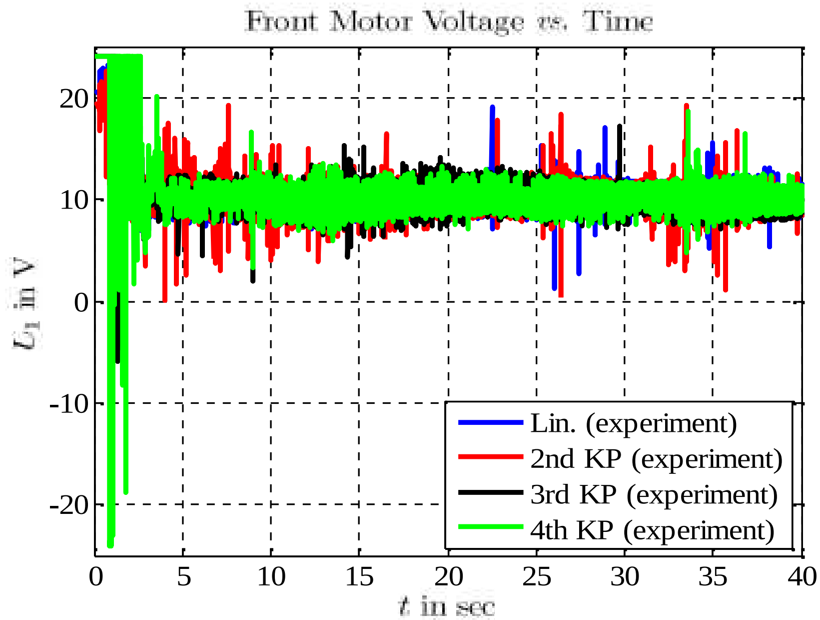

Figure G15 Front motor voltage evolution *vs.* time for desired pitch angle of sine signal of frequency 0.05 Hz and amplitude of 10 degree

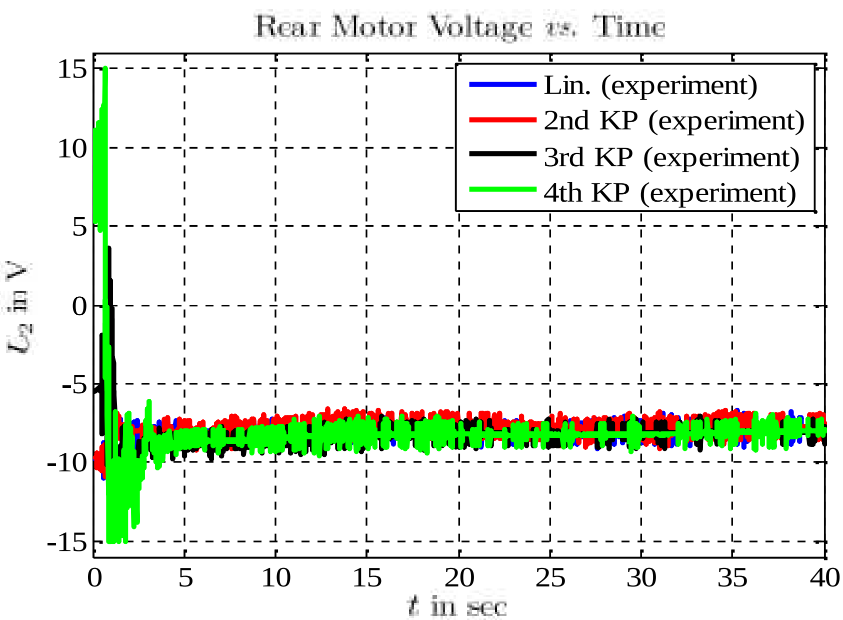

Figure G16 Rear motor voltage evolution *vs.* time for desired pitch angle of sine signal of frequency 0.05 Hz and amplitude of 10 degree

For the input controls, the experiments show that the four controllers behave in the same way except for the fourth order one which presents a higher voltage and hence energy for the front and rear motors during the start-up phase.

G.5 Sine signal desired pitch angle of 0.02 Hz frequency and 20 degree amplitude

In the following, we present the experimental results for a desired yaw angle of 0 degree, desired pitch angle of a sine signal of frequency 0.02 Hz and amplitude of 20 degree, with an initial condition of the pitch angle of -40.5 degrees for the four controllers: Linear, 2^{nd} , 3^{rd} and 4^{th} truncation orders.

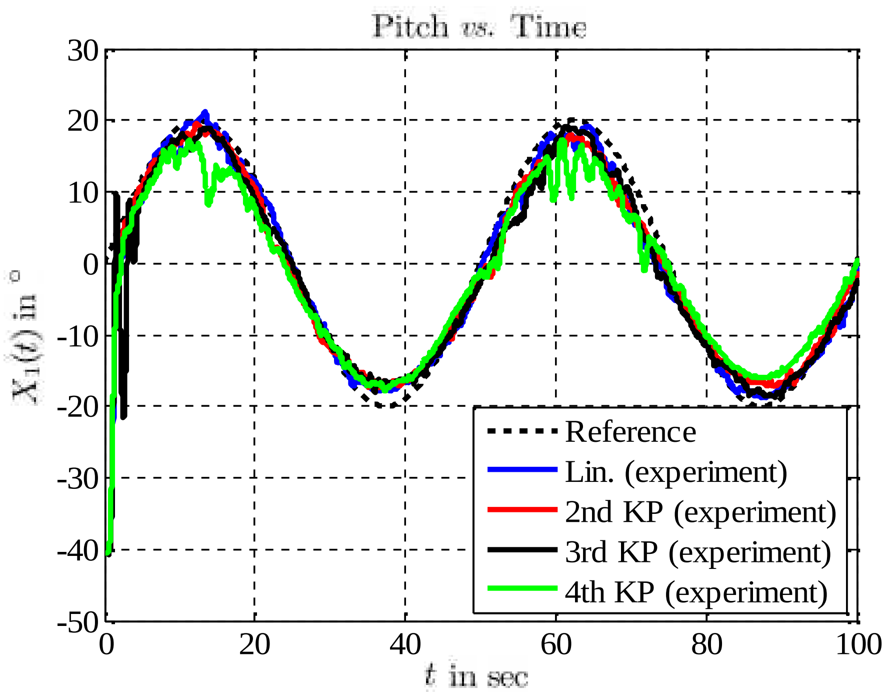

Figure G17 Pitch evolution *vs.* time for desired pitch angle of sine signal of frequency 0.02 Hz and amplitude of 20 degree

The experimental results for the pitch angle show that the four controllers stabilize the 2-DOF set-up around the desired sine pitch angle with an improvement for the second and third order controllers.

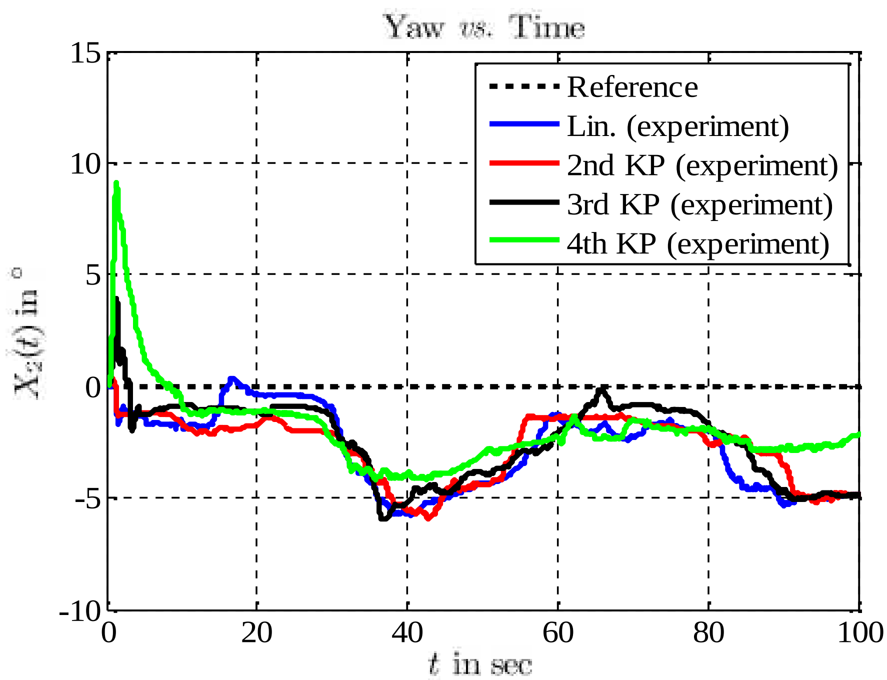

Figure G18 Yaw evolution *vs.* time for desired pitch angle of sine signal of frequency 0.02 Hz and amplitude of 20 degree

For the yaw angle, the experimental results show that the four controllers stabilize the 2-DOF set-up, with some errors about zero. We depict a high overshoot for the fourth order one.

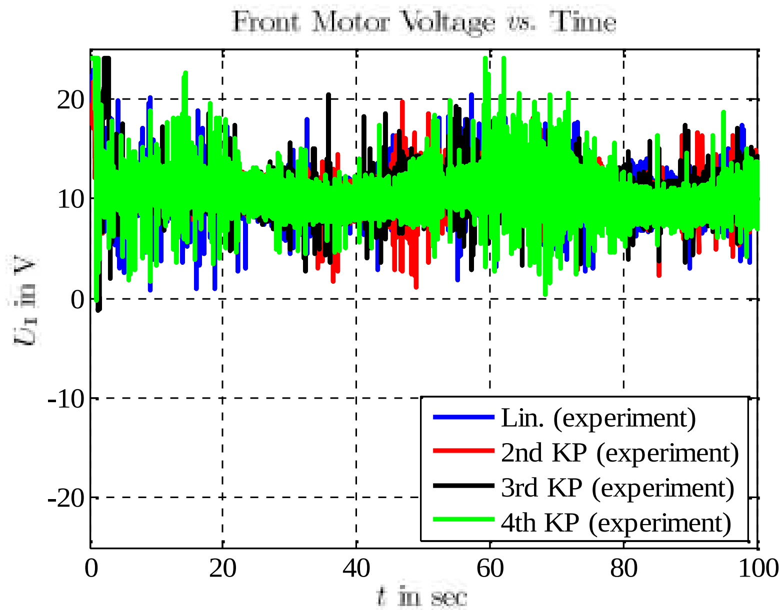

Figure G19 Front motor voltage evolution *vs.* time for desired pitch angle of sine signal of frequency 0.02 Hz and amplitude of 20 degree

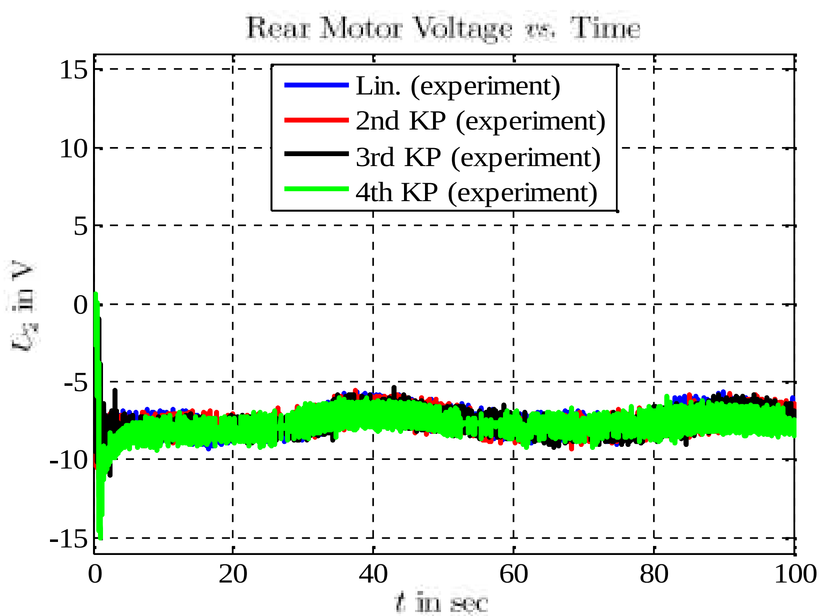

Figure G20 Rear motor voltage evolution *vs.* time for desired pitch angle of sine signal of frequency 0.02 Hz and amplitude of 20 degree

In terms of input controls, the four controllers behave almost in the same way and they present small vibrations around the equilibrium voltages of 10V for the front motor and -8V for the rear motor.

G.6 Multi-step desired pitch angle

In the following, we present the experimental results for a desired yaw angle of 0 degree, desired pitch angle of a multi-step signal, with an initial condition of the pitch angle of -40.5 degrees for the four controllers: Linear, 2^{nd}, 3^{rd} and 4^{th} truncation orders.

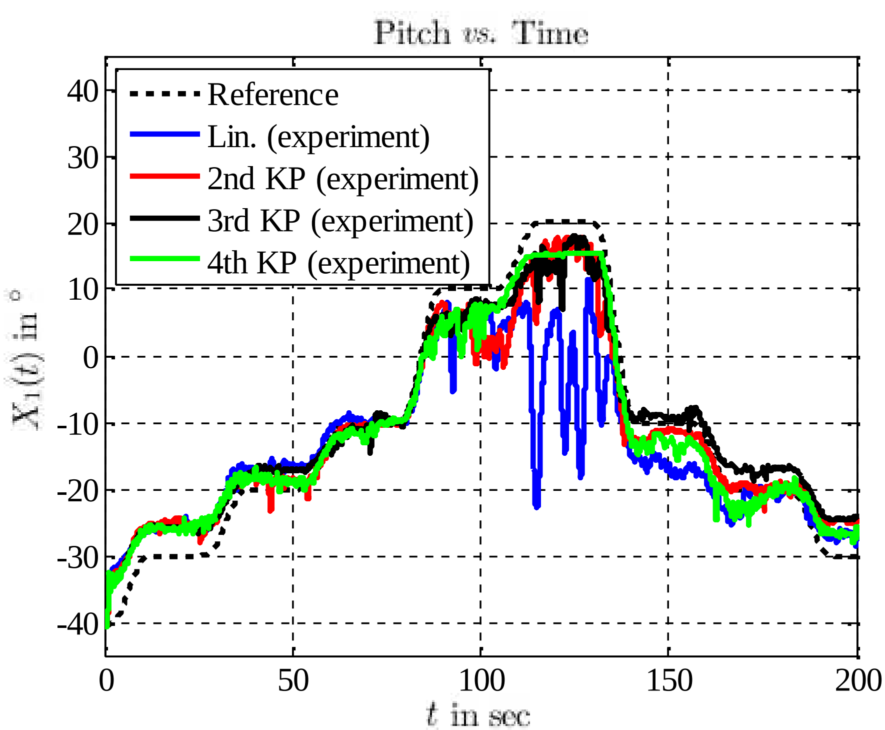

Figure G21 Pitch evolutions *vs.* time for desired pitch angle of multi-steps.

The pitch angle evolutions show better results with the 3^{rd} and 4^{th} order controllers.

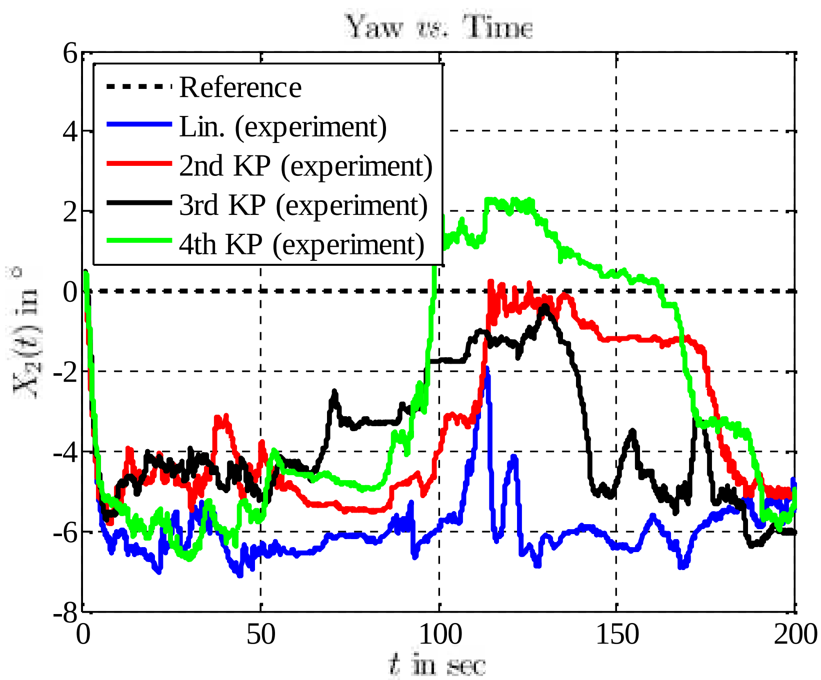

Figure G22 Yaw evolutions *vs.* time for desired pitch angle of multi-steps

The experimental results of the yaw angles for a desired pitch angle of multi-steps and a yaw angle of zero degree show that the four controllers stabilize the 2-DOF within a range of -6 to 2 degrees. The yaw performance is affected by the pitch behaviour.

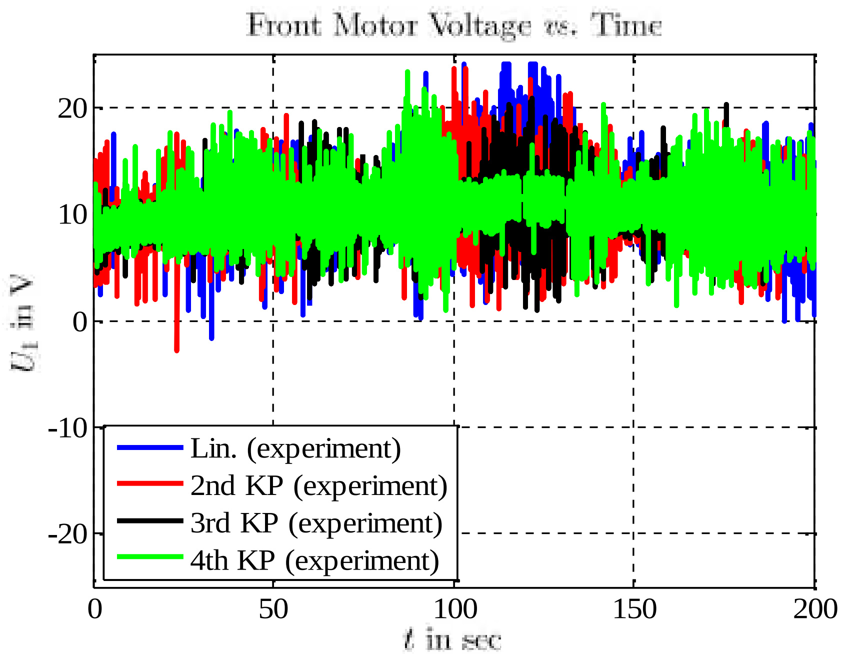

Figure G23 Front motor voltage evolution *vs.* time for desired pitch angle of multi steps

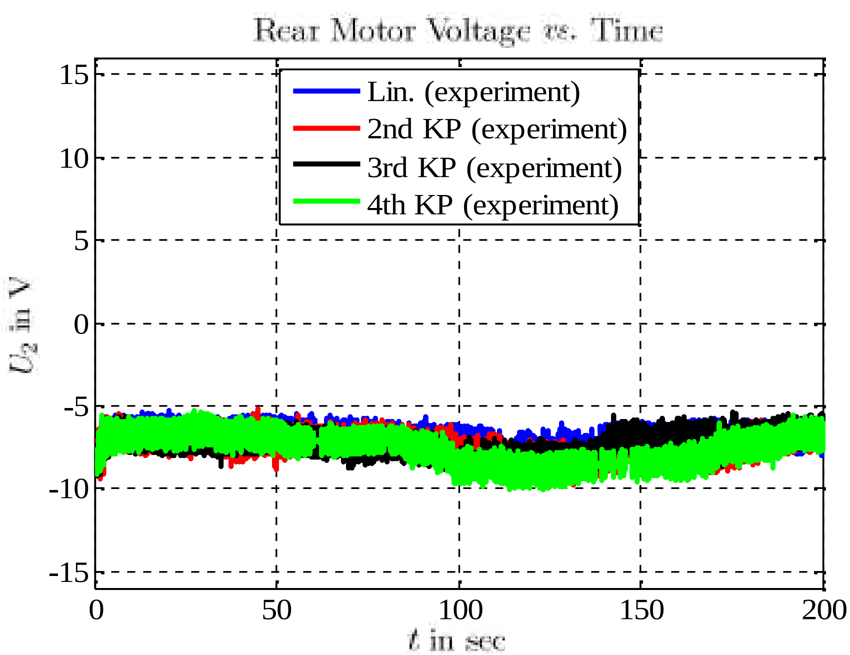

Figure G24 Rear motor voltage evolution *vs.* time for desired pitch angle of multi steps

In terms of input controls, the four controllers present almost the same behaviour and vary within a range of 5 to 20V for the front motor voltage and -10 to -6V for the rear motor voltage.

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Nonlinear Sub-Optimal Control for Polynomial Systems – Design and Stability

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Abstract– *Many real world systems are inherently nonlinear. Therefore, the linear quadratic regulator theory is rarely efficient for these systems. In this paper, we propose the design of an optimal feedback control for polynomial systems in the indeterminate state variables. To deal with the case of a nonlinear infinite-horizon-cost-functional, we investigate the control based on the Lyapunov functions (LF) and by using the Kronecker product (KP) algebra. Then, we analyze the stability of the feedback and its domain of attraction (DA) in form of convex problems based on the linear matrix inequality (LMI) formalism. The practical sub-optimal control is evaluated through simulation results and comparative schemes.*

Keywords: Polynomial Systems, Matrix KP, Nonlinear State-Feedback, Stability, Sub-Optimal Control

1. Introduction

Numerous physical systems are very well known to be nonlinear by nature, but methods for analysing and synthesizing controllers for nonlinear systems are still not as well developed as their counterparts for linear models (Ekman, 2005). The investigation of new techniques for nonlinear problems such as the stability, the estimation and the control design remains a challenge until today (see *e.g.* (Zhu & Khayati, 2012; Zhu & Khayati, 2011; Won & Biswas, 2007; Khayati et *al.*, 2006, Ekman, 2005)). In particular, to deal with the nonlinear optimal control problem, it has been stated in (Khayati, 2013) and references cited therein that a great variety of works shown in the literature used simple techniques, based on the local linearization, and more complex ones, such as (but not limited to) the state-dependent-Riccati (SDR) equation, the nonlinear-matrix-inequality- and frozen-Riccati-equation-based methods (Won & Biswas, 2007; Huang & Lu, 1996; Banks & Mhana, 1992). These methods could work well in some applications but rigorous theoretical proofs were lacking (Won & Biswas, 2007). The related grey area nevertheless covers the stability analysis of these closed loop controllers and also their implementation (complexity of the algorithms) within a large set of plants. These concerns have been discussed in separate works with a lot of compromises to achieve their goals (Won & Biswas, 2007; Ekman, 2005; Banks & Mhana, 1992).

Recently, the KP algebra has shown an important role in research activities dealing with control analysis and design (Mtar et *al.*, 2009; Bouzaouche & Braik, 2006; Rotella & Tanguy, 1988). In these works, polynomial modelling structures represent the nonlinearities using the matrix KP and the vector power algebra (Steeb, 1997; Brewer, 1978). This modelling resembles the classical linearization, but with a difference. In fact, the order of truncation of the decomposition is high enough to represent closely and fairly the actual dynamics of the system.

In this paper, the optimal control for affine input nonlinear systems (*i.e.* linear *w.r.t.* the input but nonlinear in terms of the states (Rotella & Tanguy, 1988)) is considered. Such a large class contains well-known examples in control theory and many physical systems (*e.g.* mass-spring systems with softening/hardening springs, artificial pneumatic muscles, flight engine setups, *etc.*) (Chesi, 2009; Ekman, 2005; Banks & Mhana, 1992). The controller is developed using the well-known optimality conditions (Goh 1993; Borne et *al.*, 1990; Rotella & Tanguy, 1988) by converting the nonlinear SDR equation into a set of algebraic equations using the KP algebra (Steeb, 1997; Rotella & Tanguy, 1988). The proposed method is using the same technique developed in (Rotella & Tanguy, 1988), but with a main difference of considering a given quadratic form for the cost index functional allowing the analysis of the stability of the optimal state-feedback

(Goh, 1993). In fact, this analysis will show cases where the overall system will be globally asymptotically stable (GAS), or will estimate alternatively its DA and how much this domain can be large when the system is locally asymptotically stable (LAS) eventually. The stability and DA estimate features will be cast as convex problems that will be solved using LMI frameworks (Chesi, 2009; Chesi, 2005). Indeed, we will propose a technique that ensures the computation of the largest estimation of the domain of attraction (LEDA) using both the well-known complete square matrix representation (SMR) (Chesi, 2009; Chesi, 2003) and a new formalism of a complete rectangular matrix representation (RMR).

We will proceed as follows. In Section 2, we introduce a set of useful notations, definitions and properties regarding the matrix KP algebra, the vector power series and the SMR/RMR formulations. Section 3 is devoted to the problem statement of the nonlinear dynamics, the nonlinear quadratic cost functional to be optimized and the related optimality conditions. In Section 4, we introduce an LF-based optimal cost index that will be used in the transformation of the polynomial SDR equation. Then, Section 5 deals with the computation of a 'closely' acceptable solution to this nonlinear equation in the unknown constant matrices, while in Section 6, an analytic and practical form of the state-feedback sub-optimal control is developed. Section 7 introduces the stability issue of the designed sub-optimal closed-loop. Moreover, in Section 8, we discuss the computation of the LEDA of this closed loop system. Finally, to illustrate the proposed technique, numerical and comparative results are presented in Section 9, while Section 10 concludes this work.

2. Useful Notations, Definitions and Proprieties

Notations and properties of matrices, vectors, dot product and KP tensors used in this paper are exhaustively discussed in the literature; *e.g.* (Schott, 2001; Steeb, 1997; Brewer, 1978). The proofs of the new lemmas introduced in this Section are based on theorems introduced in these references. Due to lack of space, all these theorems as well as the proofs of the lemmas shown below are omitted.

2. 1. Definitions

Definition 1: For any vector $x\in\mathbb{R}^n$ and any integer j, $x^{[j]}\in\mathbb{R}^{n^j}$ is the j-power of a vector x and $\tilde{x}^{[j]}\in\mathbb{R}^{\tau_j^{(n)}}$ is the non-redundant j-power of the vector x with $\tau_j^{(n)}$ standing for the binomial coefficient. We have $\forall j\in\mathbb{N}$, $\exists! T_j\in\mathbb{R}^{n^j\times\tau_j^{(n)}}$ *s.t.* $x^{[j]}=T_j\tilde{x}^{[j]}$ (Mtar et *al.*, 2009; Brewer, 1978).

Definition 2: Let $w(x)$ be any homogenous form of degree $2j$, then the SMR of $w(x)$ in any $x\in\mathbb{R}^n$ is given by $w(x)=\tilde{x}^{[j]T}W\tilde{x}^{[j]}$ (Chesi, 2005; Chesi, 2003). $\tilde{x}^{[j]}$ is considered a base vector of the homogenous function of degree j in x. W is a suitable but non-unique symmetric matrix SMR, also known as Gram matrix. All matrices W can be linearly parameterized as $W(\beta)=W+L(\beta)$, where $\beta\in\mathbb{R}^{\sigma(n,j)}$ is a free vector with $\sigma(n,j)=\frac{1}{2}\tau_j^{(n)}\cdot\left(\tau_j^{(n)}+1\right)-\tau_{2j}^{(n)}$. $L(\beta)\in\mathbb{R}^{\tau_j^{(n)}\times\tau_j^{(n)}}$ is a linear parameterization of the set $\left\{L=L^T \mid \tilde{x}^{[j]T}L\tilde{x}^{[j]}=0,\forall x\in\mathbb{R}^n\right\}$. We refer to $W(\beta)$ as the complete SMR of $w(x)$.

Definition 3: Let $w(x)$ any form of degree $2j+1$ in $x\in\mathbb{R}^n$ given by $w(x)=v^Tx^{[2j+1]}=x^{[2j+1]T}v$, where $v\in\mathbb{R}^{n^{2j+1}}$. Using theorem T2.13 of (Brewer, 1978), $w(x)$ can be written using a new formulation given by RMR as $w(x)=x^{[j]T}\cdot M\cdot x^{[j+1]}=x^{[j+1]T}\cdot N\cdot x^{[j]}$, with $M=mat^T_{n^j\times n^{j+1}}(v)$ and $N=mat^T_{n^{j+1}\times n^j}(v)$. Then, similarly to the homogenous forms of even order shown above, we propose a complete RMR of $w(x)$ as $\frac{1}{2}\tilde{x}^{[j]T}\left(M+L(\beta)\right)\tilde{x}^{[j+1]}+\frac{1}{2}\tilde{x}^{[j+1]}\left(M+L(\beta)\right)^T\tilde{x}^{[j]T}$, where β is a vector of free parameters. $L(\beta)\in\mathbb{R}^{\tau_j^{(n)}\times\tau_{j+1}^{(n)}}$ is a linear parameterization of the set $\left\{\tilde{x}^{[j]T}L\tilde{x}^{[j+1]}=0,\forall x\in\mathbb{R}^n\right\}$. We refer to $M(\beta)=M+L(\beta)$ as the complete RMR of $w(x)$. The following two examples illustrate this new formulation.

Example 1: Consider the form of degree 3 in two variables $w(x)=x_1^3+x_1^2x_2+x_2^3$. Noting $\tilde{x}^{[1]}=\begin{pmatrix}x_1 & x_2\end{pmatrix}^T$ and $\tilde{x}^{[2]}=\begin{pmatrix}x_1^2 & x_1x_2 & x_2^2\end{pmatrix}^T$, we obtain, for $\beta=\begin{pmatrix}\beta_1 & \beta_2\end{pmatrix}^T\in\mathbb{R}^2$,
$$M+L(\beta)=\begin{pmatrix}1 & 1+\beta_1 & \beta_2\\ -\beta_1 & \beta_2 & 1\end{pmatrix}.$$

Example 2: Consider the form of degree 3 in three variables $w(x)=x_1^3+x_1x_2x_3+x_2^3+x_2^2x_3$. Noting $\tilde{x}^{[1]}=\begin{pmatrix}x_1 & x_2 & x_3\end{pmatrix}$ and $\tilde{x}^{[2]}=\begin{pmatrix}x_1^2 & x_1x_2 & x_1x_3 & x_2^2 & x_2x_3 & x_3^2\end{pmatrix}$, we obtain, for $\beta=\begin{pmatrix}\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7\end{pmatrix}^T\in\mathbb{R}^7$, $M+L(\beta)=$
$$\begin{pmatrix}1 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5\\ -\beta_1 & -\beta_3 & 1-2\beta_4 & 1 & \beta_6 & \beta_7\\ -\beta_2 & \beta_4 & -\beta_5 & 1-\beta_6 & -\beta_7 & 0\end{pmatrix}.$$

2. 2. Notations

Notation 1: If V is a vector of dimension $p=n\cdot m$, then $M=mat_{n\times m}(V)$ is the $(n\times m)$-matrix verifying $V=vec(M)$. Therefore it is called the mat notation.

Notation 2: M^+ stands for the Moore-Penrose pseudo-inverse of any full rank matrix M.

Notation 3: Given $x \in \mathbb{R}^n$, for any integer $p \geq 1$, we denote by $X_p = \begin{pmatrix} x^{|1|T} & x^{|2|T} & \cdots & x^{|p|T} \end{pmatrix}^T$ and $\tilde{X}_p = \begin{pmatrix} \tilde{x}^{|1|T} & \tilde{x}^{|2|T} & \cdots & \tilde{x}^{|p|T} \end{pmatrix}$. We have $X_p = \mathbf{T}_p \tilde{X}_p$ where $\mathbf{T}_p \in \mathbb{R}^{N_p \times \tau_p}$ is the direct sum of T_1, T_2, ..., T_p, denoted by $\mathbf{T}_p = \bigoplus_{i=1}^{p} T_p$, with $N_p = n + n^2 + \ldots + n^p$ and $\tau_p = \tau_1^{(n)} + \tau_2^{(n)} + \cdots + \tau_p^{(n)}$ (Halmos, 1974).

Notation 4: For any vector $x \in \mathbb{R}^n$ and integers p and μ, we denote by $\chi_p^{(\mu)} = \begin{pmatrix} 1 & x^{|p|T} & \cdots & x^{|(\mu-1)p|T} \end{pmatrix}^T \in \mathbb{R}^{1+n^p+n^{2p}+\cdots+n^{(\mu-1)p}}$.

2. 3. Lemmata

Lemma 1: $\forall j \in \mathbb{N} \setminus \{0\}$ and $\forall x \in \mathbb{R}^n$ (Khayati & Benabdelkader, 2012a),

$$\frac{\partial x^{|j|}}{\partial x^T} = \mathscr{D}_j^{(n)} \cdot \left(I_n \otimes x^{|j-1|} \right) \tag{1}$$

where $\mathscr{D}_j^{(n)} \in \mathbb{R}^{n^j \times n^j}$ is given by $\mathscr{D}_j^{(n)} = \sum_{i=0}^{j-1} U_{n^i \times n} \otimes I_{n^{j-i-1}}$ and therefore called the j-differential Kronecker matrix. I_n (*resp.* $I_{n^{j-i-1}}$) denotes the identity matrix of $\mathbb{R}^{n \times n}$ (*resp.* $\mathbb{R}^{n^{j-i-1} \times n^{j-i-1}}$), $U_{n^i \times n}$ the permutation matrix of $\mathbb{R}^{n^{i+1} \times n^{i+1}}$ (Rotella & Tanguy, 1988; Brewer, 1978). Equivalently, $\mathscr{D}_j^{(n)}$ can be derived from

$$\mathscr{D}_1^{(n)} = I_n \quad \text{and} \quad \mathscr{D}_{j+1}^{(n)} = \mathscr{D}_j^{(n)} \otimes I_n + U_{n^j \times n}, \forall j \geq 1 \tag{2}$$

Lemma 2: For x and y column-vectors of $\mathbb{R}^k$ and $\mathbb{R}^l$ respectively and for any matrix $A \in \mathbb{R}^{(nk) \times l}$, we have (Khayati & Benabdelkader, 2012a)

$$\left(I_n \otimes x^T \right) Ay = \left(I_n \otimes vec^T \left(A^T \right) \right) \left(vec(I_n) \otimes I_{kl} \right) \left(x \otimes y \right) \tag{3}$$

Lemma 3: Consider a matrix $A \in \mathbb{R}^{p \times nq}$. Let $\begin{bmatrix} A_1 & \ldots & A_n \end{bmatrix}$ be a partition of A, *i.e.* $\forall i = 1, \ldots, n$, $A_i \in \mathbb{R}^{p \times q}$. We have (Khayati & Benabdelkader, 2012a)

$$\left(I_n \otimes vec^T(A) \right) \left(vec(I_n) \otimes I_{pq} \right) = mat_{pq \times n}^T \left(vec(A) \right) \tag{4}$$

3. Problem Statement

Consider the nonlinear system given by

$$\dot{x}(t) = F(x) + G(x) \cdot u(t) = F(x) + \sum_{k=1}^{m} G_k(x) \cdot u_k(t) \tag{5}$$

where $t \in \mathbb{R}$ designates the time, $x(t) \in \mathbb{R}^n$ the state vector, $u(t) = \begin{bmatrix} u_1(t) & \cdots & u_m(t) \end{bmatrix}^T \in \mathbb{R}^m$ the input vector. $F(\cdot)$ and $G_k(\cdot)$ for $k = 1, \ldots, m$ are analytic vector fields from $\mathbb{R}^n$ into $\mathbb{R}^n$ expressed as polynomials in x. Note that $G(x) = \begin{bmatrix} G_1(x) & \cdots & G_m(x) \end{bmatrix} \in \mathbb{R}^{n \times m}$. By using the KP tensor, we write $F(x) = \sum_{j=1}^{f} F_j \cdot x^{|j|}$, $\forall k = 1, \ldots, m$ $G_k(x) = \sum_{j=0}^{g} G_{kj} \cdot x^{|j|}$ and then, $G(x) = \sum_{j=0}^{g} G_j \left(I_m \otimes x^{|j|} \right)$, with $F_j \in \mathbb{R}^{n \times n^j}$, $G_{kj} \in \mathbb{R}^{n \times n^j}$ $\forall k = 1, \ldots, m$ and $G_j = \begin{bmatrix} G_{1j} & | \cdots | & G_{mj} \end{bmatrix} \in \mathbb{R}^{n \times mn^j}$. Let $z(t) = H(x) \in \mathbb{R}^q$ be a vector field in the state vector x given by $H(x) = \sum_{j=1}^{h} H_j \cdot x^{|j|}$ with $H_j \in \mathbb{R}^{q \times n^j}$ (Khayati & Benabdelkader, 2012a; Rotella & Tanguy, 1988).

For Q a symmetric non-negative definite matrix of $\mathbb{R}^{q \times q}$ and R a symmetric positive definite (SPD) matrix of $\mathbb{R}^{m \times m}$, we propose the design of a state feedback which minimizes the continuous-time cost functional

$$J = \frac{1}{2} \int_0^{\infty} \left[z(t)^T Qz(t) + u(t)^T Ru(t) \right] dt \tag{6}$$

We denote by $V(x)$ the optimal cost with an initial condition x at t (Goh, 1993; Borne et *al.*, 1990)

$$V(x) = \frac{1}{2} \int_t^{\infty} \left[z(\tau)^T Qz(\tau) + u^*(\tau)^T Ru^*(\tau) \right] d\tau \tag{7}$$

where $u^* = \arg\left(\min_u J \right)$ is the optimal control. The optimality conditions, corresponding to the problem (5) and (6), are given by (Borne et *al.*, 1990)

$$u^*(x) = -R^{-1} G(x)^T V_x(x) \tag{8}$$

$$H(x)^T QH(x) + V_x(x)^T F(x) + F(x)^T V_x(x) - V_x(x)^T G(x) \cdot V_x(x)^T G(x) R^{-1} G(x)^T V_x(x) = 0 \tag{9}$$

where $V_x(x)$ denotes the derivative of $V(x)$ *w.r.t.* the state vector x; *i.e.* $V_x(x) = \frac{\partial V}{\partial x}$.

4. Quadratic Cost Function Representation

Based on the optimality conditions discussed in (Borne et *al.*, 1990; Rotella & Tanguy, 1988), we build the following procedure to obtain a suboptimal state feedback in a polynomial form using the KP tensor, *vec* and *mat* notations (Khayati & Benabdelkader, 2012a). Such a design is based on the determination of the cost function $V(x)$ in a quadratic form. In fact, this function would be expected to satisfy the conditions of any Lyapunov candidate function (Goh, 1993). We propose (Khayati & Benabdelkader, 2012a)

$$V(x)=\frac{1}{2}\begin{pmatrix} x^T & \sum_{j=2}^{\bar{p}} x^{|j|T}\cdot P_j^T \end{pmatrix}\begin{pmatrix} P & \alpha I_n \\ \alpha I_n & I_n \end{pmatrix}\begin{pmatrix} x \\ \sum_{j=2}^{\bar{p}} P_j\cdot x^{|j|} \end{pmatrix} \quad (10)$$

with $\alpha\in\mathbb{R}$, P is an SPD constant matrix of $\mathbb{R}^{n\times n}$ and P_j constant matrices of $\mathbb{R}^{n\times n^j}$. Note that $V(x)$ can be expressed in a compact form

$$V(x)=\frac{1}{2}X_{\bar{p}}^T\mathbf{P}X_{\bar{p}} \quad (11)$$

where

$$\mathbf{P}=\begin{pmatrix} P & \alpha P_2 & \cdots & \alpha P_{\bar{p}} \\ \alpha P_2^T & P_2^T P_2 & \cdots & P_2^T P_{\bar{p}} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha P_{\bar{p}}^T & P_{\bar{p}}^T P_2 & \cdots & P_{\bar{p}}^T P_{\bar{p}} \end{pmatrix} \quad (12)$$

And equivalently, by using the Cholesky decomposition, P_1 exists *s.t.* $P=P_1^T P_1$, then the cost function $V(x)$ can be rewritten in a summation form as

$$V(x)=\frac{1}{2}\sum_{i,j=1}^{\bar{p}} x^{|i|T}P_{i(j)}^T P_{j(i)}x^{|j|} \quad (13)$$

with

$$P_{i(j)}=\begin{cases} P_1 & \text{for} \quad i=j=1 \\ \alpha I_n & \text{for} \quad i=1 \text{ and } j\geq 2 \\ P_i & \text{for} \quad i\geq 2 \text{ and } j\geq 1 \end{cases} \quad (14)$$

The expression of $V(x)$ given by (13) and (14) will be advantageous to solve the nonlinear SDR (9). Using theorems T2.3 and T4.3 in (Brewer, 1978) and applying lemmas 1, 2 and 3 and the *mat* notation, introduced in Section 2, we obtain the derivative of (13) *w.r.t.* x

$$V_x(x)=\sum_{i,j=1}^{\bar{p}}\frac{\partial x^{|j|T}}{\partial x}P_{j(i)}^T P_{i(j)}x^{|i|}=\sum_{i,j=1}^{\bar{p}}V_{ij}x^{|i+j-1|} \quad (15)$$

with

$$V_{ij}=mat_{n^{i+j-1}\times n}^T\left[vec\left(P_{i(j)}^T P_{j(i)}\mathscr{D}_j^{(n)}\right)\right] \quad (16)$$

where $\mathscr{D}_j^{(n)}$ is the square j-differential Kronecker matrix of $\mathbb{R}^{n^j\times n^j}$ introduced in lemma 1 (see Section 2). Using the KP tensor, the theorem T2.13 of (Brewer, 1978), the lemmas 2 and 3, and the *mat* notation, introduced in Section 2, we obtain from the nonlinear SDR equation (9)

$$\sum_{i,j=1}^{\bar{p}}\sum_{k=1}^{f}vec^T\left(V_{ij}^T F_k\right)x^{|i+j+k-1|}+\sum_{i,j=1}^{\bar{p}}\sum_{k=1}^{f}vec^T\left(F_k^T V_{ij}\right)x^{|i+j+k-1|}+$$
$$\sum_{i,j=1}^{h}vec^T\left(H_i^T Q H_j\right)x^{|i+j|}-\sum_{i,j,b,c=1}^{\bar{p}}\sum_{k,d=0}^{g}vec^T\left(W_{ijk}^T R^{-1}W_{bcd}\right)\cdot$$
$$x^{|i+j+k+b+c+d-2|}=0 \quad (17)$$

where

$$W_{ijk}=mat_{n^{i+j+k-1}\times m}^T\left[vec\left(V_{ij}^T G_k\right)\right] \quad (18)$$

5. Determination of P_p

In this Section, the matrices P_p, for $p=1,\ldots,\bar{p}$, will be computed from (17) by cancelling the coefficients of $x^{|p+1|}$. The details of such steps, based on the KP notations and theorems introduced in (Steeb, 1997; Brewer, 1978) as well as the lemmas 1, 2 and 3 shown in Section 2, are omitted due to lack of space.

First, the matrix P_1 is obtained by cancelling the terms of $x^{|2|}$, in (17). The operator $vec(\cdot)$ is linear on matrices of the same dimensions. Noting that the first differential Kronecker matrix is given by $\mathscr{D}_1^{(n)}=I_n$ and that $P=P_1^T P_1$ is SPD, we use (14), (16), (18) and the *mat* notation to obtain the classical algebraic Riccati equation (ARE)

$$PF_1+F_1^T P+H_1^T Q H_1-PG_0R^{-1}G_0^T P=0 \quad (19)$$

And thence, for a given $\alpha\in\mathbb{R}$, the calculation of P_p, $p=2,\ldots,\bar{p}$, is obtained from (17) by cancelling the coefficients of $x^{|p+1|}$. Using *vec* and *mat* notations, theorems T1.5, T1.6, T3.2, T3.4 of (Brewer, 1978) and the iterative form of the differential Kronecker matrix (2), we combine (14), (16) and (18) to obtain

$$\left(I_{n^{p+1}}+U_{n^p\times n}\right)\mathscr{T}_p^T\mathscr{D}_{p+1}^{(n)T}\alpha\cdot vec\left(P_p\right)=\mathscr{H}_p \quad (20)$$

where $\mathscr{T}_p = \left(F_1 - G_0 R^{-1} G_0^T P\right) \otimes I_{n^p}$ and $\mathscr{H}_p = \sum_{\substack{i,j,b,c=1 \\ i+j+k+b+c+d=p+3}}^{p-1} \sum_{k,d=0}^{p-1} vec\left(W_{ijk}^T R^{-1} W_{bcd}\right) - \sum_{\substack{i,j=1 \\ i+j+k=p+2}}^{p-1} \sum_{k=1}^{p} \left[vec\left(V_{ij}^T F_k\right) + vec\left(F_k^T V_{ij}\right)\right] - \sum_{\substack{i,j=1 \\ i+j=p+1}}^{p} vec\left(H_i^T Q H_j\right)$. Note that $\left(F_1 - G_0 R^{-1} G_0^T P\right)$ is a Hurwitz matrix, then $\mathscr{T}_p$ is regular for all $p \in \mathbb{N}$. $\mathscr{D}_{p+1}^{(n)}$ is a singular matrix for all nonzero integers p and $\left(I_{n^{p+1}} + U_{n^p \times n}\right)$ is regular for p even and singular for p odd (Khayati & Benabdelkader, 2012a; Rotella & Tanguy, 1988). Using the non-redundant vector power notation (Bouzaouche & Braiek, 2006), and the theorem T3.4 of (Brewer, 1978), we write $\tilde{P}_p = P_p \cdot T_p$ where $T_p \in \mathbb{R}^{n^p \times \tau_p^{(n)}}$ is the transformation matrix defined in Section 2 (Bouzaouche & Braiek, 2006). Two cases arise depending on p:

Case I – p is even: Let $\mathscr{T}_p = \left(T_p^+ \otimes I_n\right) \mathscr{D}_{p+1}^{(n)}$ be a full rank rectangular $\left(\left(n \cdot \tau_p^{(n)}\right) \times n^{p+1}\right)$ matrix. We obtain

$$\left(I_{n^{p+1}} + U_{n^p \times n}\right) \mathscr{T}_p^T \mathscr{T}_p^T \alpha \cdot vec\left(\tilde{P}_p\right) = \mathscr{H}_p \tag{21}$$

If P, P_2, ..., P_{p-1} are known, $\tilde{P}_p$ can be calculated as a solution of the linear equation (21). Thus, $P_p = \tilde{P}_p T_p^+$ is deduced. In fact, by using $\mathscr{T}_p^+ = \mathscr{T}_p^T \left(\mathscr{T}_p \mathscr{T}_p^T\right)^{-1}$ the Moore-Penrose pseudo-inverse of $\mathscr{T}_p$, we obtain

$$vec\left(\tilde{P}_p\right) = \frac{1}{\alpha} \mathscr{T}_p^{+T} \mathscr{T}_p^{-T} \left(I_{n^{p+1}} + U_{n^p \times n}\right)^{-1} \mathscr{H}_p \tag{22}$$

Case II – p is odd: Eq. (17) is rewritten using the non-redundant power series. Then, the coefficients of $\tilde{x}^{|p+1|}$ are given in (20), but multiplied by T_{p+1}^T on the left hand side. Thus, this linear equation becomes

$$\tilde{\mathscr{T}}_p^T \cdot vec\left(\tilde{P}_p\right) = \tilde{\mathscr{H}}_p \tag{23}$$

where $\tilde{\mathscr{T}}_p = \alpha \mathscr{T}_p \mathscr{T}_p \left(I_{n^{p+1}} + U_{n \times n^p}\right) T_{p+1}$ is a full rank rectangular $\left(\left(n \cdot \tau_p^{(n)}\right) \times \tau_{p+1}^{(n)}\right)$ matrix and $\tilde{\mathscr{H}}_p = T_{p+1}^T \cdot \mathscr{H}_p$. If P, P_2, ..., P_{p-1} are known, $\tilde{P}_p$ can be calculated as a solution of the linear equation system (23). Thus, $P_p = \tilde{P}_p T_p^+$ is deduced. In fact, by using the Moore Penrose pseudo-inverse of $\tilde{\mathscr{T}}_p$, denoted by $\tilde{\mathscr{T}}_p^+ = \left(\tilde{\mathscr{T}}_p^T \tilde{\mathscr{T}}_p\right)^{-1} \tilde{\mathscr{T}}_p^T$, we obtain

$$vec\left(\tilde{P}_p\right) = \tilde{\mathscr{T}}_p^{+T} \cdot \tilde{\mathscr{H}}_p \tag{24}$$

6. Implementation of the State Feedback

Consider the nonlinear dynamics (5). The optimal control minimizing the functional cost (6) is obtained by the optimality conditions (8) and (9). We propose the design of a practical sub-optimal control using the matrices P, P_2, ..., $P_{\bar{p}}$ computed in Section 5. It is based on an approximated optimal cost $V(x)$ given by (10). An analytical form of the state feedback can be obtained by using (8), (15), (16) and (18) (Khayati & Benabdelkader, 2012a)

$$\bar{u}(x) = -\sum_{p=1}^{\bar{p}_g} K_p x^{|p|} \tag{25}$$

with $\bar{p}_g = 2\bar{p} + g - 1$ and

$$K_p = R^{-1} \cdot \sum_{\substack{i,j=1 \\ i+j+k=p+1}}^{p} \sum_{k=0}^{g} W_{ijk} \tag{26}$$

The KP tensor is used here to design a systematic computation of a sub-optimal state-feedback. The proposed nonlinear feedback (25) with (26) would not necessarily be implemented with a great number of computed matrices P_p to be so different from the linear control approximation, *a priori*. According to (Rotella & Tanguy, 1988), it can be concluded that the state-feedback obtained with only P (*i.e.*, only the first order of the SDR equation) is more efficient than the solution issued from the linearized system. In fact, by computing only P, we may obtain a polynomial sub-optimal control of order $g+1$ (where g is the order of the term $G(x)$ in (5)), in particular, when g is non-zero. The stability of the proposed closed-loop feedback (5) and (25) will be discussed in the following section.

7. Stability of the Sub-Optimal State Feedback

To investigate the stability of the closed loop system, we consider $V(x)$, given by (10), as a Lyapunov candidate function. $V(x)$ is a radially unbounded continuous function, and its derivative exists and is continuous. From (10), if

$$\begin{pmatrix} P & \alpha I_n \\ \alpha I_n & I_n \end{pmatrix} > 0 \tag{27}$$

holds, then the Lyapunov candidate function $V(x)$ is positive definite; that is $V(x)>0$, $\forall x\neq 0$. Note that (27) is equivalent to $P>\alpha^2 I_n$. The time derivative of the LF $V(x)$, along the trajectories of the closed loop system (5) and (25), is given by

$$\dot{V}(x)=V(x)^T\cdot\dot{x}(t)=V(x)^T F(x)-\bar{u}(x)^T R\bar{u}(x) \tag{28}$$

Let us define B_1 and C_1 by $G_0R^{-1}G_0^T$ and $H_1^TQH_1$, respectively. We assume the triplet (F_1,B_1,C_1) is stabilizable-detectable. Note that if a solution P of the ARE (19) exists, then it is the unique SPD matrix solution of the optimal control for the linearized system and (F_1-B_1P) is a Hurwitz matrix (Rotella & Tunguy, 1988). Thus, the linearized system is asymptotically stable. Moreover, the nonlinear closed loop system (5) and (25) is LAS and $\exists x\neq 0$ *s.t.* $\frac{\partial(x^TPx)}{\partial t}<0$.

In the following, we assume $\{x\in\mathbb{R}^n\setminus\{0\}\mid\dot{V}<0\}\neq\varnothing$ and consider the closed ball $\mathcal{B}(\delta)=\{x\in\mathbb{R}^n\mid\|x\|\leq\delta\}$. Given α *s.t.* $P>\alpha^2I_n$; *i.e.* $V(x)>0$ for all nonzero $x\in\mathbb{R}^n$, $\mathcal{B}(\delta)$ is an estimate of the DA if $\mathcal{B}(\delta)\subset\Delta=\{x\in\mathbb{R}^n\mid\dot{V}<0\}\cup\{0\}$ (Chesi, 2009; Chesi, 2003). The computation of the maximum δ *s.t.* $\mathcal{B}(\delta)\subset\Delta$, *i.e.* (5) and (25) is LAS, corresponds to the LEDA of the closed-loop dynamics and is given by $\mathcal{B}(\gamma)$ where (Chesi, 2009)

$$\gamma=\inf_{x\in\mathbb{R}^n\setminus\{0\}\ s.t.\ \dot{V}(x)=0}\|x\| \tag{29}$$

8. LEDA Computation of the Closed Loop System

In this section, we present the mechanism to evaluate the LEDA of the obtained sub-optimal closed-loop system. Let $\mathcal{S}(\delta)=\{x\in\mathbb{R}^n\mid\|x\|=\delta\}$ be a given sphere. The problem (29) turns out that (Chesi, 2003)

$$\gamma=\sup\{\bar{\delta}\mid\dot{V}(x)<0,\forall x\in\mathcal{S}(\delta),\forall\delta\in(0,\bar{\delta}]\} \tag{30}$$

We assume that P, P_2, ..., $P_{\bar{p}}$ are obtained from (19), (21) and (23). The terms $V(x)^TF(x)$ and $\bar{u}(x)^TR\bar{u}(x)$ are polynomials in x of degrees $2\bar{p}+f-1$ and $2\bar{p}_g$, respectively. For any $\delta>0$, we have $\forall x\in\mathcal{S}(\delta)$

$$\dot{V}(x)=\sum_{\substack{k=1\\k+l=p}}^{2\bar{p}-1}\sum_{l=1}^{f}x^{|k|T}v_k^TF_lx^{|l|}-\sum_{\substack{i,j=1\\i+j=p}}^{\bar{p}_g}x^{|i|T}K_i^TRK_jx^{|j|} \tag{31}$$

with $v_k=\sum_{\substack{i,j=1\\i+j-1=k}}^{\bar{p}}V_{ij}$, where V_{ij} is given by (16). Using the non-redundant vector power series $\tilde{x}^{|p|}$ and the vector notations $X_{\bar{p}}$ introduced in Section 2, without loss of generality, we assume that $\exists\bar{p}_t\in\mathbb{N}$, with $0<\bar{p}_t\leq\bar{p}_g$, and $\exists\Gamma_t=\Gamma_t^T>0$ *s.t.* $\sum_{i,j=1}^{\bar{p}_g}x^{|i|T}K_i^TRK_jx^{|j|}=X_{\bar{p}_t}^T\Gamma_tX_{\bar{p}_t}+w_e(x)+w_o(x)$. The terms $w_e(x)$ and $w_o(x)$ are polynomials in even and odd vector powers in x of orders $2\bar{p}_e$ and $2\bar{p}_o+1$, respectively. $\bar{p}_e$ and $\bar{p}_o$ are integers *s.t.* $0\leq\bar{p}_e\leq\bar{p}_g$ and $0\leq\bar{p}_o\leq\bar{p}_g$. Then, we use the SMR and RMR notations introduced in Section 2 to set the time derivative of the LF, $\dot{V}(x)$, in a quadratic form. If we denote by $\bar{p}_q=\max\left(\bar{p}_e,\bar{p}+\frac{f-1}{2}\right)$ and $\bar{p}_r=\max\left(\bar{p}_o,\bar{p}+\frac{f-3}{2}\right)$ for f odd, and $\bar{p}_q=\max\left(\bar{p}_e,\bar{p}+\frac{f}{2}-1\right)$ and $\bar{p}_r=\max\left(\bar{p}_o,\bar{p}+\frac{f}{2}-1\right)$ for f even, respectively. We obtain

$$\dot{V}(x)=-\sum_{i=1}^{\bar{p}_q}\tilde{x}^{|i|T}S_{ii}\left(\beta_i^{(e)}\right)\tilde{x}^{|i|}-\sum_{i=1}^{\bar{p}_r}\left[\tilde{x}^{|i|T}S_{i,i+1}\left(\beta_i^{(o)}\right)\tilde{x}^{|i+1|}+\tilde{x}^{|i+1|T}\cdot S_{i,i+1}^T\left(\beta_i^{(o)}\right)\tilde{x}^{|i|}\right]-X_{\bar{p}_t}^T\Gamma_tX_{\bar{p}_t} \tag{32}$$

where $S_{ii}\left(\beta_i^{(e)}\right)\in\mathbb{R}^{\tau_i^{(n)}\times\tau_i^{(n)}}$ is the SMR matrix of the terms of order $2i$ in $\sum_{k=1}^{2\bar{p}-1}\sum_{l=1}^{f}x^{|k|T}v_k^TF_lx^{|l|}+w_e(x)$, $\beta_i^{(e)}\in\mathbb{R}^{\sigma(n,i)}$ a free vector with $\sigma(n,i)=\frac{1}{2}\tau_i^{(n)}.\left(\tau_i^{(n)}+1\right)-\tau_{2i}^{(n)}$ and $\tau_i^{(n)}$ stands for binomial coefficients (Mtar et *al.*, 2009), and $S_{i,i+1}\left(\beta_i^{(o)}\right)$ is the RMR of terms of order $2i+1$ in $\frac{1}{2}\sum_{k=1}^{2\bar{p}-1}\sum_{l=1}^{f}x^{|k|T}v_k^TF_lx^{|l|}+\frac{1}{2}w_o(x)$ (see Section 2). (32) can be rewritten as follows

$$\dot{V}(x)=-\tilde{X}_{\bar{p}_s}^T\mathbf{S}(\beta)\tilde{X}_{\bar{p}_s}-X_{\bar{p}_t}^T\Gamma_tX_{\bar{p}_t} \tag{33}$$

where $\bar{p}_s=\max(\bar{p}_q,\bar{p}_r)$ and

$$\mathbf{S}(\beta)=\begin{pmatrix}S_{11}&S_{12}&0&0&\dots&0\\S_{12}^T&S_{22}&S_{23}&0&\dots&0\\0&S_{23}^T&S_{33}&\ddots&&\vdots\\0&0&\ddots&\ddots&&0\\\vdots&\vdots&&&&\\0&0&&0&&\end{pmatrix} \tag{34}$$

The decision variables β are set by the concatenation of all free variables $\beta_i^{(e)}$ and $\beta_i^{(o)}$, $\forall i$. Two cases arise depending on the size of the values of $\bar{p}_s$ and $\bar{p}_t$ in (33).

8. 1. Case of $\bar{p}_s = \bar{p}_t$

Using the transformation $X_{\bar{p}_t} = \mathbf{T}_{\bar{p}_t} \tilde{X}_{\bar{p}_t}$ introduced in Section 2, we have $\dot{V}(x) < 0$ if the LMI

$$\mathbf{S}(\beta) + \mathbf{T}_{\bar{p}_t}^T \Gamma_t \mathbf{T}_{\bar{p}_t} > 0 \tag{35}$$

holds in the free decision variable β. Thus, for P solution of the ARE (19), given α *s.t.* $P > \alpha^2 I_n$ and P_2, ..., $P_{\bar{p}}$ computed from (21) and (23), if the LMI (35) problem is feasible in β, then the sub-optimal state-feedback (5) and (25) is GAS.

8. 2. Case of $\bar{p}_s \neq \bar{p}_t$

Let ν be the least common multiple of $\bar{p}_s$ and $\bar{p}_t$, *i.e.* $\exists(\nu_s, \nu_t) \in \mathbb{N}^2 \setminus \{(0,0)\}$ *s.t.* $\nu = \nu_s p_s = \nu_t p_t$. Consider the well-posed vectors $\chi_{p_s}^{(\nu_s)}$ and $\chi_{p_t}^{(\nu_t)}$ introduced in Section 2. Noting $\forall x \in \mathbb{S}(\delta)$, $\|x\| = \delta$, then we have $\left\|\chi_{\bar{p}_s}^{(\nu_s)}\right\|^2 = \sum_{i=0}^{\nu_s - 1} \delta^{2i\bar{p}_s} = \delta_s^2$, $\left\|\chi_{\bar{p}_t}^{(\nu_t)}\right\|^2 = \sum_{i=0}^{\nu_t - 1} \delta^{2i\bar{p}_t} = \delta_t^2$ and $\chi_{\bar{p}_s}^{(\nu_s)} \otimes X_{\bar{p}_s} = \chi_{\bar{p}_t}^{(\nu_t)} \otimes X_{\bar{p}_t} = X_\nu$. Thus, (33) is equivalent to

$$\dot{V}(x) = -X_\nu^T \left(\frac{1}{\delta_s^2} I_{\xi_s} \otimes \mathbf{S}^+(\beta) + \frac{1}{\delta_t^2} I_{\xi_t} \otimes \Gamma_t \right) X_\nu \tag{36}$$

with $\mathbf{S}^+(\beta) = \mathbf{T}_{\bar{p}_s}^{+T} \mathbf{S}(\beta) \mathbf{T}_{\bar{p}_s}^+$. $\mathbf{T}_{\bar{p}_s}^+$ is the pseudo-inverse of $\mathbf{T}_{\bar{p}_s} \in \mathbb{R}^{N_{\bar{p}_s} \times \tau_{\bar{p}_s}}$ introduced in Section 2, $\xi_s = 1 + n^{\bar{p}_s} + n^{2\bar{p}_s} + \cdots + n^{(\nu_s - 1)\bar{p}_s}$, $\xi_t = 1 + n^{\bar{p}_t} + n^{2\bar{p}_t} + \cdots + n^{(\nu_t - 1)\bar{p}_t}$. Noting that Γ_t is SPD, let $\mathbf{T}_{\bar{p}_t}^{+T} \Gamma_t \mathbf{T}_{\bar{p}_t}^+ = \bar{\Theta}^T \bar{\Theta}$ be the Cholesky decomposition. Then, from (36), $\forall x \in \mathbb{S}(\delta)$, $\dot{V}(x) < 0$ is equivalent to

$$\left(I_{\xi_t} \otimes \bar{\Theta}^{-T}\right)\left(I_{\xi_s} \otimes \mathbf{S}^+(\beta)\right)\left(I_{\xi_t} \otimes \bar{\Theta}^{-1}\right) + \bar{\delta} \cdot I_{\xi_t \cdot N_{\bar{p}_t}} > 0 \tag{37}$$

where the factor $\bar{\delta} = \delta_s^2 / \delta_t^2$ depends on δ. If $\nu_s > \nu_t \Leftrightarrow \bar{p}_s < \bar{p}_t$, then $\bar{\delta} > 1$ and monotically increasing with δ. If $\nu_s < \nu_t \Leftrightarrow \bar{p}_s > \bar{p}_t$, then $\bar{\delta} < 1$ and monotically decreasing with δ. The following results hold.

Sub-case $\bar{p}_s < \bar{p}_t$*:* $\forall \beta$, $\exists \bar{\delta} > 1$ *s.t.* the LMI (37) holds. Thus, for P solution of the ARE (19), given α *s.t.* $P > \alpha^2 I_n$ and P_2, ..., $P_{\bar{p}}$ computed from (21) and (23), the sub-optimal state-feedback system (5) and (25) is GAS.

Sub-case $\bar{p}_s > \bar{p}_t$*:* Given $\upsilon \in \mathbb{R}$, consider the LMI

$$\left(I_{\xi_t} \otimes \bar{\Theta}^{-T}\right)\left(I_{\xi_s} \otimes \mathbf{S}^+(\beta)\right)\left(I_{\xi_t} \otimes \bar{\Theta}^{-1}\right) - \upsilon I_{\xi_t \cdot N_{\bar{p}_t}} > 0 \tag{38}$$

in the vector β and the scalar υ. If $\exists \upsilon \geq 0$ *s.t.* the LMI (38) holds, then the LMI constraint (37) holds $\forall \bar{\delta} > 0$, then we select $\bar{\delta} < 1$ and we have $\bar{\delta}$ decreasing with δ (*i.e.* $\delta \to \infty$ as $\bar{\delta} \to 0$). Thus, for P solution of the ARE (19), given α *s.t.* $P > \alpha^2 I_n$ and P_2, ..., $P_{\bar{p}}$ computed from (21) and (23), if the LMI (38) is feasible in $\upsilon \geq 0$ and β, then the sub-optimal state-feedback system (5) and (9) is GAS.

Sub-case $\bar{p}_s > \bar{p}_t$ *and* $\exists \upsilon$ *s.t.* $-1 < \upsilon < 0$*:* A lower bound $\bar{\gamma}$ of γ, given by (30), is computed by $\bar{\gamma} = \arg_\delta \dfrac{1 + \delta^{2\bar{p}_s} + \cdots + \delta^{2(\nu_s - 1)\bar{p}_s}}{1 + \delta^{2\bar{p}_t} + \cdots + \delta^{2(\nu_t - 1)\bar{p}_t}} = (-\bar{\upsilon})$, where $\bar{\upsilon}$ is a solution of the following eigen-value problem (EVP): $\bar{\upsilon} = \max \upsilon$ subject to $-1 < \upsilon < 0$ and LMI (38). If $\arg\max_\upsilon$ of this EVP is negative, then the linear inequality constraint $-1 < \upsilon < 0$ corresponds to $\bar{\delta} < 1$ as $\bar{p}_s > \bar{p}_t$.

Remark: The results discussed above can be proven using simply the theorem 1 of (Chesi, 2003) and the proposition 2 of (Chesi, 2005).

9. Example

As an example, we consider the design of a nonlinear aircraft flight control problem which has been exhaustively treated in literature (see *e.g.* (Banks & Mhana, 1992)) and defined by

$$\begin{aligned}
\dot{x}_1 &= -0.877x_1 + x_3 - 0.088x_1x_3 + 0.47x_1^2 - 0.019x_2^2 - x_1^2x_3 + \\
&\quad -0.215u + 0.28x_1^2u + 0.47x_1u^2 + 0.63u^3 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -4.208x_1 - 0.396x_3 - 0.47x_1^2 - 3.564x_1^3 - 20.967u + \\
&\quad 6.265x_1^2u + 46x_1u^2 + 61.4u^3
\end{aligned}$$

where x_1 is the angle of attack in rad, x_2 the pitch angle in rad, x_3 the pitch rate in rad/sec and u the control input provided by the tail deflection angle in rad (Banks & Mhana, 1992). Note that terms involving nonlinearities in u with small effect on the dynamics are eliminated, as the approaches discussed here cannot account for nonlinear control terms, but are taken into consideration in the simulations. The performance index uses $H(x) = x$, $Q =$ $0.25 \cdot I_3$ and $R = 1$. The simulations have been applied for the proposed 'LF'-based technique as well as the linear control 'Lin' where the dynamics is linearized about the origin, the 'KP'-based design introduced in (Rotella & Tanguy, 1988) and the SDR-equation-pointwise-based (referred to as 'PW') technique (Banks & Mhana, 1992). The sub-optimal cost J is evaluated with different initial conditions in terms of angle of attack, $x_1(0)$, and same $x_2(0)=x_3(0)=0$ for the different methods. Table 1 shows the cost performance errors $\varepsilon_J^{\bullet} = \frac{J_{\text{pw}} - J_{\bullet}}{J_{\text{pw}}}$ in $\%$. The 'LF'- (of orders 2 and 3), 'KP'- (of orders 2 and 3) and 'Lin'-based design costs are compared to the 'PW'-technique one. A positive value corresponds to an improvement (*i.e.*, a lower cost) with the given method compared to the 'PW' cost, meanwhile a negative value corresponds to a higher cost. Figures 1-3 show the control variable, the angle of attack and the pitch angle, respectively, obtained with the initial condition $x_1(0)=23^{\circ}$. Due to lack of space the pitch rate figure is omitted. Curves of 'LF'-based design, with orders of truncation 2 and 3, overlap almost during all the time showing very similar results in terms of transient behaviour and stability. Furthermore, the proposed design (with both orders 2 and 3 which are relatively small) exhibits a significant added-value in terms of cost estimation and domain of attraction interval performances compared to the other methods.

Table 1. Cost index J^{PW} and cost errors (expressed in $\%$ of J^{PW}) $\varepsilon_{J(p=2)}^{\text{LF}}$, $\varepsilon_{J(p=3)}^{\text{LF}}$, $\varepsilon_{J(p=2)}^{\text{KP}}$, $\varepsilon_{J(p=3)}^{\text{KP}}$, $\varepsilon_{J}^{\text{Lin}}$.

$x_1(0)$	J^{PW}	$\varepsilon_{J(p=2)}^{\text{LF}}$	$\varepsilon_{J(p=3)}^{\text{LF}}$	$\varepsilon_{J(p=2)}^{\text{KP}}$	$\varepsilon_{J(p=3)}^{\text{KP}}$	$\varepsilon_{J}^{\text{Lin}}$
6°	0.0016	**20.2**	**18.6**	-0.6	-0.8	0.0
12°	0.0071	**23.8**	**22.8**	-1.6	-2.6	-0.2
17°	0.0196	**30.9**	**30.3**	-3.7	-6.8	-0.7
23°	0.0519	**46.3**	**45.7**	-13.3	-31.7	-4.3
29°	0.1056	**48.3**	**46.3**	Unstab.	Unstab.	Unstab.
34°	0.4081	**71.4**	**65.6**	Unstab.	Unstab.	Unstab.
40°	1.6170	**58.5**	**50.9**	Unstab.	Unstab.	Unstab.

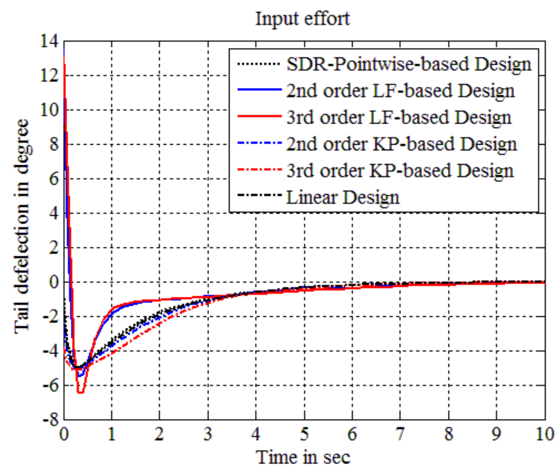

Fig. 1. Input control *vs.* time.

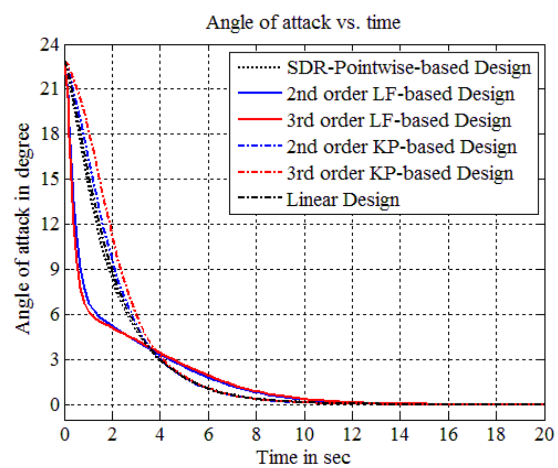

Fig. 2. Angle of Attack *vs.* time

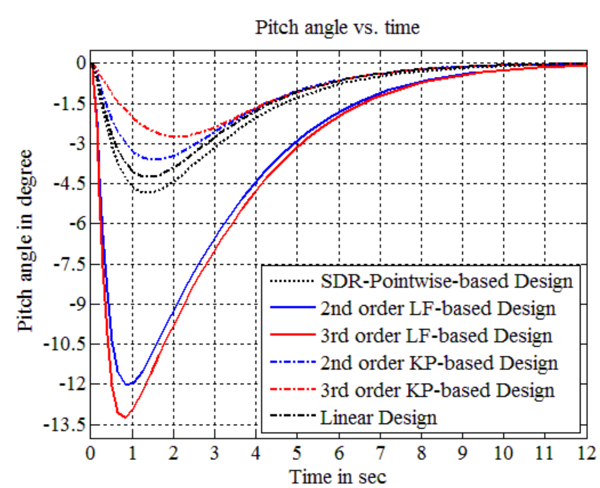

Fig. 3. Pitch Angle *vs.* time

10. Conclusions

A new nonlinear optimal control design for polynomial systems subject to nonlinear cost objectives is proposed. We develop a systematic and practical LF-based sub-optimal control approach using the KP notations. The analysis of the stability of the closed loop system is then discussed using LMI frameworks. The problem of the LEDA computation is cast as a convex EVP design. This method is expected to ensure a best compromise between the feasibility of the implemented scheme and the stability analysis of the overall system. An example showing simulations and comparative results successfully demonstrates the effectiveness of this technique. Furthermore, a modified version of this nonlinear optimal control will be presented to relax the conditions within the computation of the Lyapunov function matrices of high order, and also, improving the formulation of the stability feature (Khayati, 2013). Nevertheless, all those changes will be proposed by following the same overall procedure discussed in this paper.

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Nonlinear Sub-Optimal Control for Polynomial Systems - New Design

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Abstract -Many real world systems are inherently nonlinear and therefore the linear quadratic regulator theory is rarely efficient for these systems. In this paper, we propose the design of an optimal feedback control for nonlinear systems expressed as formal vector power series (VPS) in the indeterminate state variables. The problem of an infinite horizon with a nonlinear cost function is investigated based on the Lyapunov function (LF) design and using the Kronecker product (KP) algebra. The proposed scheme represents a key element for a follow-on work discussing the stability of the given nonlinear state feedback. A practical sub-optimal control is evaluated through simulations.

***Keywords*:** Polynomial systems, KP, Nonlinear optimal control.

1. Introduction

Numerous physical systems are very well known to be nonlinear by nature, and various control problems need to be treated within nonlinear concepts in order to deal with their complexity maturely and efficiently (Zhu and Khayati, 2011; Khayati et *al*., 2006; Ekman 2005). Thence, it still remains a challenge to investigate new analytic rules and alternative numerical techniques for nonlinear problems such as stability, control design and optimal control (Won and Biswas, 2007; Ekman, 2005). In particular, the nonlinear optimal control design has been a popular subject for a number of researchers, but there are still issues to overcome therein. Indeed, a great variety of work exists in the literature using simple techniques based on local linearization or more complex ones, such as (but not limited to) the state-dependent-Ricatti (SDR) equation, nonlinear-matrix-inequality-based and frozen-Riccati-equation-based methods (Won and Biswas, 2007; Huang and Lu, 1996; Banks and Mhana, 1992). These methods seem to work well in some applications but rigorous theoretical proofs are very weak regarding the stability of the closed loop design, which is rarely globally asymptotically stable (GAS), and also the implementation, due to the complexity of the algorithm (Won and Biswas, 2007). These concerns are often discussed in separate works with less compromise (Won and Biswas, 2007; Ekman, 2005; Banks and Mhana, 1992).

In this paper, the nonlinear optimal control of a quadratic cost function with higher order terms applied to an affine control nonlinear system (that is linear in control action but nonlinear in the states) is considered to propose a practical state-feedback. Such a large class contains well-known examples in control theory and many physical systems; *e.g.* mass-spring systems with softening/hardening springs, artificial pneumatic muscles, flight engine setups, *etc.* (Chesi, 2009; Ekman, 2005; Banks and Mhana, 1992). The optimal controller is calculated using the well-known optimality conditions discussed in (Goh 1993; Borne et *al*., 1990; Rotella and Tanguy, 1988) by converting the given nonlinear Hamilton-Jacobi equation (HJE) into a system of algebraic equations through the KP algebra introduced in (Steeb, 1997; Brewer, 1978). This method is using the same technique developed in (Rotella and Tanguy, 1988), but with a difference of considering a given quadratic form for the cost index function leading to the stability conditions of the optimal state-feedback as discussed in the (Goh, 1993). In Section 2, some properties that are useful for the present work will be introduced. Section 3 is devoted to the problem statement of the nonlinear dynamics, the nonlinear quadratic cost function to be optimised and the related optimality conditions. In Section 4, the LF-based optimal cost that will be used in the transformation of the nonlinear polynomial HJE equation is discussed. Section 5 is devoted to the computation of a solution to this nonlinear equation in the unknown constant matrices based on the proposed KP and VPS decomposition. In Section 6, an analytic and practical form of the state-feedback optimal control is discussed. Finally, numerical and comparative results are presented in Section 7 to illustrate the proposed technique, while Section 8 concludes this work.

2. Useful Proprieties and Notations

Notations and properties of matrices, vectors, dot product and KP tensors used in this paper are exhaustively discussed in the literature (Steeb, 1997; Brewer, 1978). In the following, we limit our presentation to new lemma and a given notation. The proofs of these lemmas are based on theorems introduced in (Brewer, 1978). Due to lack of space, all notations and theorems useful for this work and also the proofs of the following lemmas are omitted (and remain available upon request).

Lemma 1: $\forall\, j \in \mathbb{N} \setminus \{0\}$ and $\forall x \in \mathbb{R}^n$,

$$\frac{\partial x^{[j]}}{\partial x^T} = \mathscr{D}_j^{(n)} \cdot \left(I_n \otimes x^{[j-1]} \right) \tag{1}$$

where $\mathscr{D}_j^{(n)} \in \mathbb{R}^{n^j \times n^j}$ is given by $\mathscr{D}_j^{(n)} = \sum_{i=0}^{j-1} U_{n^i \times n} \otimes I_{n^{j-i-1}}$ and therefore called the square j-differential Kronecker matrix. $I_{n^{j-i-1}}$ denotes the identity matrix of $\mathbb{R}^{n^{j-i-1}}$, $U_{n^i \times n}$ the permutation matrix of $\mathbb{R}^{n^{i+1} \times n^{i+1}}$ defined in (Rotella and Tanguy, 1988; Brewer, 1978). Equivalently, $\mathscr{D}_j^{(n)}$ can be derived from

$$\begin{cases} \mathscr{D}_1^{(n)} = I_n \\ \mathscr{D}_{j+1}^{(n)} = \mathscr{D}_j^{(n)} \otimes I_n + U_{n^j \times n}, \forall j \geq 1 \end{cases} \tag{2}$$

Lemma 2: For x and y any column-vectors of $\mathbb{R}^k$ and $\mathbb{R}^l$ respectively and for any matrix $A \in \mathbb{R}^{nk \times l}$, we have

$$\left(I_n \otimes x^T \right) Ay = \left(I_n \otimes vec^T\left(A^T\right)\right)\left(vec\left(I_n\right) \otimes I_{kl} \right)\left(x \otimes y \right) \tag{3}$$

Lemma 3: Consider a matrix $A \in \mathbb{R}^{p \times nq}$. Let $\left[A_1 \quad \ldots \quad A_n \right]$ be a partition of A, *i.e.* $\forall i = 1, \ldots, n$ $A_i \in \mathbb{R}^{p \times q}$. We have

$$\left(I_n \otimes vec^T\left(A\right)\right)\left(vec\left(I_n\right) \otimes I_{pq} \right) = mat^T_{pq \times n}\left(vec\left(A\right)\right) \tag{4}$$

Notation: If V is a vector of dimension $p = n \cdot m$, then $M = mat_{n \times m}\left(V\right)$ is the $\left(n \times m\right)$-matrix verifying $V = vec\left(M\right)$. Therefore it is called the mat notation.

3. Problem Statement

3. 1. Nonlinear Dynamics and Nonlinear Optimal Objective Function

Consider the polynomial system given by

$$\dot{x}(t)=F(x)+G(x)\cdot u(t)=F(x)+\sum_{k=1}^{m}G_k(x)\cdot u_k(t) \tag{5}$$

where $t\in\mathbb{R}$ designates the time, $x(t)\in\mathbb{R}^n$ the state vector, $u(t)=\begin{bmatrix}u_1(t) & \cdots & u_m(t)\end{bmatrix}^T\in\mathbf{R}^m$ the input vector. $F(\cdot)$, $G_k(\cdot)$ for $k=1,\dots,m$ are analytic vector fields from $\mathbf{R}^n$ into $\mathbf{R}^n$. By using the KP tensor and the VPS decomposition, we denote by $F(x)=\sum_{j=1}^{f}F_j\cdot x^{|j|}$, $\forall k=1,\dots,m$ $G_k(x)=\sum_{j=0}^{g}G_{kj}\cdot x^{|j|}$ and then, $G(x)=\begin{bmatrix}G_1(x) & \cdots & G_m(x)\end{bmatrix}=\sum_{j=0}^{g}G_j\left(I_m\otimes x^{|j|}\right)$, with $F_j\in\mathbf{R}^{n\times n^j}$, $G_{kj}\in\mathbf{R}^{n\times n^j}$ $\forall k=1,\dots,m$, and $G_j=\begin{bmatrix}G_{1j} & |\cdots| & G_{mj}\end{bmatrix}\in\mathbf{R}^{n\times mn^j}$. Let $z(t)=H(x)\in\mathbf{R}^q$ be a vector function of the states, where $H(x)=\sum_{j=1}^{h}H_j\cdot x^{|j|}$ with $H_j\in\mathbf{R}^{q\times n^j}$.

For Q a symmetric non-negative definite matrix of $\mathbf{R}^{q\times q}$ and R a symmetric positive definite (SPD) matrix of $\mathbf{R}^{m\times m}$, the optimal control problem is to design a state feedback which minimizes the continuous-time cost functional

$$J=\frac{1}{2}\int_0^{\infty}\left[z(t)^T Qz(t)+u(t)^T Ru(t)\right]dt \tag{6}$$

3. 2. Optimality Condition

We denote by $V(x)$ the optimal cost with an initial condition x at t

$$V(x)=\frac{1}{2}\int_t^{\infty}\left[z(\tau)^T Qz(\tau)+u^*(\tau)^T Ru^*(\tau)\right]d\tau \tag{7}$$

where $u^*=\arg\left(\min_u J\right)$ is the optimal control. The optimality conditions are given by (Borne et *al.*, 1990):

$$u^*(x)=-R^{-1}G(x)^T V_x(x) \tag{8}$$

$$H(x)^T QH(x)+V_x(x)^T F(x)+F(x)^T V_x(x)-V_x(x)^T G(x)R^{-1}G(x)^T V_x(x)=0 \tag{9}$$

4. Alternative for the Nonlinear HJE Equation

To find the cost function $V(x)$ satisfying the conditions of any Lyapunov candidate function as discussed in literature (Goh, 1993), we propose the following quadratic form

$$V(x)=\frac{1}{2}\begin{pmatrix}x^T & \sum_{j=2}^{\bar{p}}x^{|j|T}\cdot P_j^T\end{pmatrix}\begin{pmatrix}P & \alpha I_n\\ \alpha I_n & I_n\end{pmatrix}\begin{pmatrix}x\\ \sum_{j=2}^{\bar{p}}P_j\cdot x^{|j|}\end{pmatrix} \tag{10}$$

with $\alpha \in \mathbf{R}$, P is an SPD constant matrix of $\mathbf{R}^{n\times n}$ and P_j constant matrices of $\mathbf{R}^{n\times n^j}$. Note that using Cholesky decomposition, P_1 exists *s.t.* $P = P_1^T P_1$. The cost function $V(x)$ can be rewritten in a summation form as

$$V(x) = \frac{1}{2}\sum_{i,j=1}^{\bar{p}} x^{|i|T} P_{i(j)}^T P_{j(i)} x^{|j|} \tag{11}$$

with

$$P_{i(j)} = \begin{cases} P_1 & \text{for} \quad i = j = 1 \\ \alpha I_n & \text{for} \quad i = 1 \text{ and } j \geq 2 \\ P_i & \text{for} \quad i \geq 2 \text{ and } j \geq 1 \end{cases} \tag{12}$$

Using theorems T2.3 and T4.3 in (Brewer, 1978) and applying lemmas 1, 2 and 3 and the *mat* notation, introduced in section 2, we obtain the derivative of (12) *w.r.t.* x

$$V_x(x) = \sum_{i=1}^{\bar{p}}\sum_{j=1}^{\bar{p}} \frac{\partial x^{|j|T}}{\partial x} P_{j(i)}^T P_{i(j)} x^{|i|} = \sum_{i=1}^{\bar{p}}\sum_{j=1}^{\bar{p}} V_{ij} x^{|i+j-1|} \tag{13}$$

with

$$V_{ij} = mat_{n^{i+j-1}\times n}^T \left[vec\left(P_{i(j)}^T P_{j(i)} \mathscr{D}_j^{(n)} \right) \right] \tag{14}$$

where $\mathscr{D}_j^{(n)}$ is the square j-differential Kronecker matrix of $\mathbf{R}^{n^j\times n^j}$ introduced in lemma 1. Introducing the KP tensor into the nonlinear HJE equation (9) and using theorem T2.13 in (Brewer, 1978), lemmas 2 and 3, and the *mat* notation, introduced in section 2, we obtain

$$\sum_{i,j=1}^{\bar{p}}\sum_{k=1}^{f} vec^T\left(V_{ij}^T F_k\right) x^{|i+j+k-1|} + \sum_{i,j=1}^{\bar{p}}\sum_{k=1}^{f} vec^T\left(F_k^T V_{ij}\right) x^{|i+j+k-1|} + \sum_{i,j=1}^{h} vec^T\left(H_i^T Q H_j\right) x^{|i+j|} - \sum_{i,j,b,c=1}^{\bar{p}}\sum_{k,d=0}^{g} vec^T\left(W_{ijk}^T R^{-1} W_{bcd}\right) x^{|i+j+k+b+c+d-2|} = 0 \tag{15}$$

where

$$W_{ijk} = mat_{n^{i+j+k-1}\times m}^T \left[vec\left(V_{ij}^T G_k \right) \right] \tag{16}$$

5. Determination of P_p

The matrices P_p, for $p = 1,\ldots,\bar{p}$, will be calculated from (15) by cancelling the coefficients of $x^{|p+1|}$. The details of such steps, based on KP, *vec* and *mat* notations and theorems introduced in (Steeb, 1997; Brewer, 1978) as well as lemmas 1, 2 and 3 shown in Section 2, are omitted due to lack of space.

5.1. First order

The matrix P_1 is obtained by cancelling the terms of $x^{|2|}$, in (15). The operator $vec(\cdot)$ is linear on matrices of the same dimensions. Noting that the 1^{st} differential Kronecker matrix $\mathcal{D}_1^{(n)} = I_n$ and that $P = P_1^T P_1$ is SPD, we use (12), (14), (16) and the mat notation to obtain the classical algebraic Ricatti equation

$$PF_1 + F_1^T P + H_1^T Q H_1 - PG_0 R^{-1} G_0^T P = 0 \tag{17}$$

5.2. Higher order

For a given $\alpha \in \mathbb{R}$, the calculation of P_p, $p = 2, \ldots, \overline{p}$, is obtained from (15) by cancelling the coefficients of $x^{|p+1|}$. Using vec and mat notations, theorems T1.5, T1.6, T3.2 and T3.4 of (Brewer, 1978), and the iterative form of the differential Kronecker matrix (3), we combine (12), (14) and (16) to obtain

$$\left(I_{n^{p+1}} + U_{n^p \times n}\right) \mathcal{F}_p^T \mathcal{D}_{p+1}^{(n)T} \cdot vec\left(P_p\right) = \mathcal{H}_p \tag{18}$$

where $\mathcal{F}_p = \left(F_1 - G_0 R^{-1} G_0^T P\right) \otimes I_{n^p}$ and $\mathcal{H}_p = \sum_{\substack{i,j,b,c=1 \\ i+j+k+b+c+d=p+3}}^{p-1} \sum_{k,d=0}^{p-1} vec\left(W_{ijk}^T R^{-1} W_{bcd}\right) - \sum_{\substack{i,j=1 \\ i+j+k=p+2}}^{p-1} \sum_{k=1}^{p} \left[vec\left(V_{ij}^T F_k\right) + vec\left(F_k^T V_{ij}\right)\right] - \sum_{\substack{i,j=1 \\ i+j=p+1}}^{p} vec\left(H_i^T Q H_j\right)$. Note that $\left(F_1 - G_0 R^{-1} G_0^T P\right)$ is a Hurwitz matrix (Rotella, 1988), $\mathcal{D}_{p+1}^{(n)}$ is a singular matrix for all integers $p = 2, \ldots, \overline{p}$ and $\left(I_{n^{p+1}} + U_{n^p \times n}\right)$ is regular for p even and singular for p odd. Using the non-redundant vector power notation $\tilde{x}^{|p|}$ introduced in (Bouzaouache and Braiek, 2006), and the theorem T3.4 in (Brewer, 1978), we write $\tilde{P}_p = P_p \cdot T_p$ where $T_p \in \mathbb{R}^{n^p \times \tau_p^{(n)}}$ with $\tau_p^{(n)}$ stands for the binomial coefficient, and thus $P_p = \tilde{P}_p T_p^+$. Note that for any integer p, $T_p \in \mathbb{R}^{n^p \times \tau_p^{(n)}}$ exists, *s.t.* $x^{|p|} = T_p \tilde{x}^{|p|}$ and $\tilde{x}^{|p|} = T_p^+ x^{|p|}$ where $T_p^+ = \left(T_p^T T_p\right)^{-1} T_p^T$ is the Moore-Penrose pseudo-inverse of T_p (Bouzaouache and Braiek, 2006). Two cases arise depending on p:

5.2.1. Case I – p is even: Let $\mathcal{T}_p = \left(T_p^+ \otimes I_n\right) \mathcal{D}_{p+1}^{(n)}$ be a rectangular $\left(\left(n \cdot \tau_p^{(n)}\right) \times n^{p+1}\right)$-matrix of full rank. We obtain

$$\left(I_{n^{p+1}} + U_{n^p \times n}\right) \mathcal{F}_p^T \mathcal{T}_p^T \cdot vec\left(\tilde{P}_p\right) = \mathcal{H}_p \tag{19}$$

By using $\mathcal{T}_p^+ = \mathcal{T}_p^T \left(\mathcal{T}_p \mathcal{T}_p^T\right)^{-1}$ the Moore-Penrose pseudo-inverse of $\mathcal{T}_p$, we obtain

$$vec\left(\tilde{P}_p\right) = \mathcal{T}_p^{+T} \mathcal{F}_p^{-T} \left(I_{n^{p+1}} + U_{n^p \times n}\right)^{-1} \mathcal{H}_p \tag{20}$$

If P, P_2, …, P_{p-1} are known, $\tilde{P}_p$ can be calculated as a solution of the linear equation (20). Thus, $P_p = \tilde{P}_p T_p^+$ is deduced.

5.2.2. Case II – p is odd: Eq. (15) can be written in terms of non-redundant vector power $\tilde{x}^{|j|}$. Then the coefficients of $\tilde{x}^{|p+1|}$ are given in (20), but multiplied by T_{p+1}^{T} on the left hand side. Then by the use of the non-redundant form, this linear equation becomes

$$\tilde{\mathscr{F}}_p^{T} \cdot vec\left(\tilde{P}_p\right) = \tilde{\mathscr{H}}_p \qquad (21)$$

Where $\tilde{\mathscr{F}}_p = \mathscr{T}_p \mathscr{F}_p \left(I_{n^{p+1}} + U_{n \times n^p}\right) T_{p+1}$ and $\tilde{\mathscr{H}}_p = T_{p+1}^{T} \cdot \mathscr{H}_p$, with $\tilde{\mathscr{F}}_p$ is a rectangular $\left(\left(n \cdot \tau_p^{(n)}\right) \times \tau_{p+1}^{(n)}\right)$-matrix of full rank. By using $\tilde{\mathscr{F}}_p^{+} = \tilde{\mathscr{F}}_p^{T}\left(\tilde{\mathscr{F}}_p \tilde{\mathscr{F}}_p^{T}\right)^{-1}$ the Moore Penrose pseudo-inverse of $\tilde{\mathscr{F}}_p$, we obtain

$$vec\left(\tilde{P}_p\right) = \tilde{\mathscr{F}}_p^{+T} \cdot \tilde{\mathscr{H}}_p \qquad (22)$$

If P, P_2, …, P_{p-1} are known, $\tilde{P}_p$ can calculated as a solution of the linear equation system (22). Thus, $P_p = \tilde{P}_p T_p^{+}$ is deduced.

6. Implementation of the State Feedback

Consider the nonlinear dynamics (5). The optimal control which minimizes the functional cost (6) is obtained by the optimality conditions (8) and (9). We propose to use the procedure introduced in section 4 and 5 with the approximated optimal cost $V(x)$ of (10). To solve the obtained nonlinear HJE equation (9), transformed in the form of (15), it was shown that the cancellation of the $\bar{p}$ first terms $x^{|2|}$, $x^{|3|}$, …, $x^{|\bar{p}+1|}$ leads to independent equations in P, P_2, …, $P_{\bar{p}}$, respectively. We propose to construct a practical suboptimal control of the analytical expression $\bar{u}(x)$ by using (8), (13) and (16).

$$\bar{u}(x) = -\sum_{p=1}^{2\bar{p}+g-1} K_p x^{|p|} \qquad (23)$$

with

$$K_p = R^{-1} \cdot \underbrace{\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p-1}}_{i+j+k=p+1} W_{ijk} \qquad (24)$$

Thus, the use of KP algebra allows a systematic determination of a sub-optimal state-feedback. The proposed nonlinear feedback (23) with (24) has not to be necessarily implemented with a high order of computed matrices P_p to be so different from the linear control approximation, *a priori*. According to (Rotella and Tanguy, 1988), it can be concluded that the stat-feedback obtained with only P (*i.e.*, only the first order of the HJE equation) is more efficient than the solution issued from the linearized system. In fact, by computing only P, we may obtain a polynomial sub-optimal control of order $g+1$ (where g is the order of the term $G(x)$ in (5)), in particular, when g is non-zero. The stability of the proposed state feedback (24) will be discussed in a further work (see (Khayati, 2012)) by considering $V(x)$ given by (10) as a Lyapunov candidate function.

7. Example

As an example, we consider the design of a nonlinear aircraft flight control problem which has been extensively treated in literature (see *e.g.* (Banks, 1992)) defined by $\dot{x}_1 = -0.877x_1 + x_3 - 0.088x_1x_3 + 0.47x_1^2 - 0.019x_2^2 - x_1^2x_3 + 3.846x_1^3 - 0.215u + 0.28x_1^2u + 0.47x_1u^2 + 0.63u^3$, $\dot{x}_2 = x_3$ and $\dot{x}_3 = -4.208x_1 - 0.396x_3 - 0.47x_1^2 - 3.564x_1^3 - 20.967u + 6.265x_1^2u + 46x_1u^2 + 61.4u^3$, where x_1 is the angle of attack in rad, x_2 the pitch angle in rad, x_3 the pitch rate in rad/sec, u the control input provided by the tail deflection angle in rad. Note that terms involving nonlinearities in u with small effect on the dynamics are eliminated, as the approaches discussed here cannot account for nonlinear control terms (Banks, 1992), but are taken into consideration in the simulation. The performance index is minimized using $H(x) = x$, $Q = 0.25 \cdot I_3$ and $R = 1$. The simulations have been applied for the proposed 'LF'-based technique as well as the linear control 'Lin' where the dynamics is linearized about the origin, the 'KP'-based design introduced in (Rotella and Tanguy, 1988) and an SDR-equation-pointwise-based technique (Banks, 1992) (referred to as 'pw' in the following). The sub-optimal cost $J_{\square}$ is evaluated with different initial conditions in terms of angle of attack, $x_1(0)$ given in degree, but with the same initial conditions $x_2(0) = x_3(0) = 0$, *vs.* the different methods. Table 1 shows the cost performance errors $\varepsilon_J^{\square} = \dfrac{J_{\text{pw}} - J_{\square}}{J_{\text{pw}}}$ in $\%$; where the 'LF'- based design (with $p = 2$ and $p = 3$), the 'KP'-based design (of orders 2 and 3) and the 'Lin'-based design costs are compared to the 'pw'-technique one. A positive value corresponds to an improvement (*i.e.*, a lower cost) with the given method compared to the 'pw' one; meanwhile the negative value means a higher cost. Figure 1 shows the angle of attack and the control variable, respectively, obtained with the initial condition $x_1(0) = 23^{\circ}$. Due to lack of space the pitch and pitch rate figures are omitted. Curves of 'LF'-based design with orders of truncation of $p = 2$ and $p = 3$ overlap almost during all the time showing very similar results in terms of transient behaviour and stability. Furthermore, the proposed design (with both orders $p = 2$ and $p = 3$ which remain relatively small) exhibits a significant added-value in terms of cost estimation and domain of attraction interval performances compared to the other methods.

Table. 1. Cost index J^{pw} and cost errors (in $\%$ of J^{KP}) $\varepsilon_{J(p=2)}^{\text{LF}}$, $\varepsilon_{J(p=3)}^{\text{LF}}$, $\varepsilon_{J(p=2)}^{\text{KP}}$, $\varepsilon_{J(p=3)}^{\text{KP}}$, $\varepsilon_J^{\text{Lin}}$

$x_1(0)$	J^{pw}	$\varepsilon_{J(p=2)}^{\text{LF}}$	$\varepsilon_{J(p=3)}^{\text{LF}}$	$\varepsilon_{J(p=2)}^{\text{KP}}$	$\varepsilon_{J(p=3)}^{\text{KP}}$	$\varepsilon_J^{\text{Lin}}$
6°	0.0016	**20.2**	**18.6**	–0.6	–0.8	0.0
12°	0.0071	**23.8**	**22.8**	–1.6	–2.6	–0.2
17°	0.0196	**30.9**	**30.3**	–3.7	–6.8	–0.7
23°	0.0519	**46.3**	**45.7**	–13.3	–31.7	–4.3
29°	0.1056	**48.3**	**46.3**	Unstable	Unstable	Unstable
34°	0.4081	**71.4**	**65.6**	Unstable	Unstable	Unstable
40°	1.6170	**58.5**	**50.9**	Unstable	Unstable	Unstable

8. Conclusion

A practical nonlinear optimal control design for nonlinear dynamics subject to nonlinear cost objectives is proposed. An example with simulation and comparative results successfully demonstrates the effectiveness of this technique. This method is developed in order to ensure a best compromise between the feasibility of the implemented scheme and the stability analysis of the overall system; which will be the subject of a follow-on work.

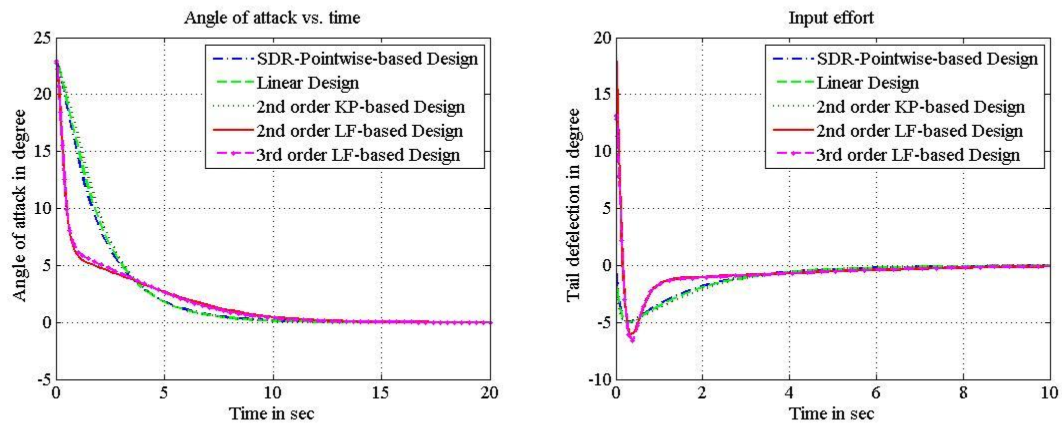

Fig. 1. Angle of attack *vs.* time (left) and Input control *vs.* time (right)

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Nonlinear Sub-Optimal Control for Polynomial Systems – Stability Analysis

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***Abstract* –** This paper presents an investigation on the stability performance and the domain of attraction (DA) of a Lyapunov function (LF) based infinite horizon nonlinear optimal control design. A polynomial modeling structure represents the nonlinearities using the Kronecker Product (KP) algebra and the vector power series (VPS). A practical scheme is proposed to reduce the complexity of such nonlinear design and to improve the requirements in terms of stability and bounds of the DA. The computation of the control parameters is partially based on a linear matrix inequality (LMI) feasibility problem. Furthermore the asymptotic stability analysis and the DA estimate are cast as convex problems.

***Keywords*:** Polynomial systems, KP, Optimal control, Asymptotic stability, Lyapunov function, LMI.

1. Introduction

Many real world systems are inherently nonlinear, but methods for analyzing and synthesizing controllers for nonlinear systems are still not as well developed as their counterparts (Ekman, 2005). Recently, the LF-based control has been generating renewed interest in nonlinear optimal control (Won and Biswas, 2007). Also, the KP algebra has an important role in recent researches dealing with control analysis and design (Mtar et *al.*, 2009; Bouzaouche and Braik, 2006; Rotella and Tanguy, 1988). A polynomial modeling structure represents the nonlinearities using the KP and VPS algebra (Steeb, 1997; Brewer, 1978). This modeling resembles classical linearization, but with a difference. In fact, the order of truncation of the decomposition is high enough to represent well (closely and fairest possible) the actual dynamics of the system. In (Khayati and Benabdelkader, 2012), we have discussed the design and implementation of a practical scheme for nonlinear control based on LF and KP concepts. We have transformed the Hamilton-Jacobi equation (HJE) into a set of algebraic equations in the matrices elements of the quadric candidate LF. The infinite horizon control design is derived in formal power series in the indeterminate state variables.

The objective of this work is to evaluate the stability of the overall system, to estimate its DA and eventually to enlarge this domain. The case of the globally asymptotically stable (GAS) state-feedback for the nonlinear optimal control problem is definitely discussed. This investigation is led under a set of convex problems that will be solved using the LMI frameworks (Chesi, 2009; Chesi, 2005). We propose a technique that ensures the computation of the largest estimation of the domain of attraction (LEDA) using both the well-known complete square matrix representation (SMR) (Chesi, 2009; Chesi, 2003) and a new formalism of a complete rectangular matrix representation (RMR). In Section 2, we introduce a set of useful notations and definitions; in particular definitions of the existing SMR and a new RMR. Then, we recall, in Section 3, the problem statement and the key element of the sub-optimal control design discussed in (Khayati and Benabdelkader, 2012). Section 4 is devoted to the sub-optimal state feedback stability features. In Section 5, we discuss the computation of the LEDA of the closed loop system. The evaluation of a numeric simulated scalar system expected to be shown is omitted due to lack of space (readers can still refer to the optimal control of third order dynamics performances discussed in (Khayati and Benabdelkader, 2012) in terms of stability and DA), while Section 6 presents concluding remarks.

2. Useful Notations and Definitions

Notations and properties of matrices, vectors, dot product and KP tensors used in this paper are exhaustively discussed in the literature (Steeb, 1997; Brewer, 1978) and are not shown here due to lack of space (but remain available upon request).

- For any vector $x \in \mathbb{R}^n$ and any integer j, $x^{|j|}$ is the j-power of a vector x and $\tilde{x}^{|j|}$ is the non-redundant j-power of the vector x (Steeb, 1997; Brewer, 1978). We have: $\forall j \in \mathbb{N}$, $\exists! T_j \in \mathbb{R}^{n^j \times \tau_j^{(n)}}$ *s.t.* $x^{|j|} = T_j \tilde{x}^{|j|}$ and $\tilde{x}^{|j|} = T_j^+ x^{|j|}$ with $T_j^+ = \left(T_j^T T_j\right)^{-1} T_j^T$ the Moore-Penrose pseudo-inverse of T_j. $\tau_j^{(n)}$ stands for the binomial coefficients (Mtar et *al.*, 2009; Bouzaouache and Braiek, 2006; Brewer, 1978).
- For any $p \geq 1$, we denote by $X_p = \left(x^{|1|T} \quad x^{|2|T} \quad \cdots \quad x^{|p|T}\right)^T$ and $\tilde{X}_p = \left(\tilde{x}^{|1|T} \quad \tilde{x}^{|2|T} \quad \cdots \quad \tilde{x}^{|p|T}\right)$.

We have $X_p = \mathbf{T}_p \tilde{X}_p$ and $\tilde{X}_p = \mathbf{T}_p^+ X_p$ where $\mathbf{T}_p = \begin{pmatrix} T_1 & 0 & \cdots & 0 \\ 0 & T_2 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & T_p \end{pmatrix} \in \mathbb{R}^{N_p \times \tau_p}$ and

$\mathbf{T}_p^+ = \begin{pmatrix} T_1^+ & 0 & \cdots & 0 \\ 0 & T_2^+ & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & T_p^+ \end{pmatrix}$ with $N_p = n + n^2 + \ldots + n^p$ and $\tau_p = \tau_1^{(n)} + \tau_2^{(n)} + \cdots + \tau_p^{(n)}$.

- For any vector $x \in \mathbb{R}^n$ and integers p and μ, we denote by $\chi_p^{(\mu)} = \left(1 \quad x^{|p|T} \quad \cdots \quad x^{|(\mu-1)p|T}\right)^T \in \mathbb{R}^{1+n^p+n^{2p}+\cdots+n^{(\mu-1)p}}$.
- If V is a vector of dimension $p = n \cdot m$, then $M = mat_{n\times m}(V)$ is the $(n \times m)$-matrix verifying $V = vec(M)$. Therefore, it is called the mat notation.
- Let $w(x)$ be any homogenous form of degree $2j$, then the SMR of $w(x)$ is defined as: $w(x) = \tilde{x}^{|j|T} W \tilde{x}^{|j|}$, where $\tilde{x}^{|j|} \in \mathbb{R}^{\tau_j^{(n)}}$ is considered a base vector of the homogenous function of degree j in any $x \in \mathbb{R}^n$ and W is a suitable but non-unique symmetric matrix SMR (Chesi, 2003; Reznick, 2003). All matrices W can be linearly parameterized as: $W(\beta) = W + L(\beta)$, where $\beta \in \mathbb{R}^{\sigma(n,j)}$ is a free vector and $\sigma(n,j) = \frac{1}{2}\tau_j^{(n)} \cdot \left(\tau_j^{(n)} + 1\right) - \tau_{2j}^{(n)}$ with $\tau_j^{(n)}$ stands for binomial coefficients. $L(\beta) \in \mathbb{R}^{\tau_j^{(n)} \times \tau_j^{(n)}}$ is a linear parameterization of the set $\left\{L = L^T \mid \tilde{x}^{|i|T} L \tilde{x}^{|i|} = 0, \forall x \in \mathbb{R}^n\right\}$. We refer to $W(\beta)$ as the complete SMR of $w(x)$.
- Let $w(x)$ any form of degree $2j+1$ in any $x \in \mathbb{R}^n$ given by $w(x) = v^T x^{|2j+1|} = x^{|2j+1|T} v$, where $v \in \mathbb{R}^{n^{2j+1}}$. Using theorem T2.13 of (Brewer, 1978), $w(x)$ can be written using a new formulation given by RMR as: $w(x) = x^{|j|T} \cdot M \cdot x^{|j+1|} = x^{|j+1|T} \cdot N \cdot x^{|i|}$, with $M = mat^T_{n^j \times n^{j+1}}(v)$ and $N = mat^T_{n^{j+1} \times n^j}(v)$. Then, similarly to the homogenous forms of even order, we propose a complete RMR of $w(x)$ as:

$w(x)=\frac{1}{2}\tilde{x}^{|j|T}\left(M+L(\beta)\right)\tilde{x}^{|j+1|}+\frac{1}{2}\tilde{x}^{|j+1|}\left(M+L(\beta)\right)^T\tilde{x}^{|j|T}$, where β is a vector of free parameters. $L(\beta)\in\mathbb{R}^{\tau_j^{(n)}\times\tau_{j+1}^{(n)}}$ is a linear parameterization of the set $\left\{\tilde{x}^{|j|T}L\tilde{x}^{|j+1|}=0,\forall x\in\mathbb{R}^n\right\}$. We refer to $M(\beta)=M+L(\beta)$ (*resp.* $M(\beta)^T$) as the complete RMR of $w(x)$.

Example 1: Consider the form of degree 3 in two variables $w(x)=x_1^3+x_1^2x_2+x_2^3$. We have $\tilde{x}^{|1|}=\begin{pmatrix}x_1 & x_2\end{pmatrix}^T$ and $\tilde{x}^{|2|}=\begin{pmatrix}x_1^2 & x_1x_2 & x_2^2\end{pmatrix}^T$. We obtain $M+L(\beta)=\begin{pmatrix}1 & 1+\beta_1 & \beta_2\\ -\beta_1 & \beta_2 & 1\end{pmatrix}$ and $\beta=\begin{pmatrix}\beta_1 & \beta_2\end{pmatrix}^T\in\mathbb{R}^2$.

Example 2: Consider the form of degree 3 in three variables $w(x)=x_1^3+x_1x_2x_3+x_2^3+x_2^2x_3$. We have $\tilde{x}^{|1|}=\begin{pmatrix}x_1 & x_2 & x_3\end{pmatrix}$ and $\tilde{x}^{|2|}=\begin{pmatrix}x_1^2 & x_1x_2 & x_1x_3 & x_2^2 & x_2x_3 & x_3^2\end{pmatrix}$. We obtain $M+L(\beta)=$

$$M+L(\beta)=\begin{pmatrix}1 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5\\ -\beta_1 & -\beta_3 & 1-2\beta_4 & 1 & \beta_6 & \beta_7\\ -\beta_2 & \beta_4 & -\beta_5 & 1-\beta_6 & -\beta_7 & 0\end{pmatrix} \text{ and } \beta=\begin{pmatrix}\beta_1\\ \vdots\\ \beta_7\end{pmatrix}\in\mathbb{R}^7.$$

3. Problem statement

3. 1. Nonlinear Dynamics and Nonlinear Optimal Objective Function

Consider the polynomial system given by:

$$\dot{x}(t)=F(x)+G(x)\cdot u(t)=F(x)+\sum_{k=1}^{m}G_k(x)\cdot u_k(t) \tag{1}$$

where $t\in\mathbb{R}$ is the time, $x(t)\in\mathbb{R}^n$ the state vector, $u(t)=\left[u_1(t) \;\cdots\; u_m(t)\right]^T\in\mathbf{R}^m$ the input vector. $F(.)$, $G_k(.)$, $\forall k=1,...,m$ are analytic vectors fields from $\mathbf{R}^n$ into $\mathbf{R}^n$, given by the polynomial forms $F(x)=\sum_{j=1}^{f}F_j\cdot x^{|j|}$, $G_k(x)=\sum_{j=0}^{g}G_{kj}\cdot x^{|j|}$ and then $G(x)=\left[G_1(x) \;|\cdots|\; G_m(x)\right]$ $=\sum_{j=0}^{g}G_j\left(I_m\otimes x^{|j|}\right)$. Let $z(t)=H(x)\in\mathbb{R}^q$ a vector function of the states with $H(x)=\sum_{j=1}^{h}H_j\cdot x^{|j|}$. Note $F_j\in\mathbb{R}^{n\times n^j}$, $G_{kj}\in\mathbb{R}^{n\times n^j}$ $\forall k=1,\cdots,m$ and $G_j=\left[G_{1j} \;|\cdots|\; G_{mj}\right]\in\mathbb{R}^{n\times mn^j}$ (Khayati and Benabdelkader, 2012).

For Q a symmetric non-negative definite matrix of $\mathbf{R}^{q\times q}$ and R a symmetric positive definite (SPD) matrix of $\mathbf{R}^{m\times m}$, the optimal control problem is to design a state feedback which minimizes the continuous-time cost functional

$$J=\frac{1}{2}\int_0^{\infty}\left[z(t)^TQz(t)+u(t)^TRu(t)\right]dt \tag{2}$$

If we denote by $u^* = \arg\left(\min_u J\right)$ the optimal control, the optimal cost $V\left(x\right)$, with an initial condition x at t, is

$$V\left(x\right)=\frac{1}{2}\int_t^{\infty}\left[z\left(\tau\right)^T Qz\left(\tau\right)+u^*\left(\tau\right)^T Ru^*\left(\tau\right)\right]d\tau \tag{3}$$

3. 2. Lyapunov-Function-Based Sub-Optimal Controller

Based on the optimality conditions introduced in literature (see *e.g.* (Borne et *al.*, 1990; Rotella and Tanguy, 1988)), we have built a procedure to obtain a sub-optimal state feedback in a polynomial form using the KP tensor, vec and mat notations (Khayati and Benabdelkader, 2012). Such a design is based on the determination of the cost function $V\left(x\right)$ in a quadratic form, as

$$V\left(x\right)=\frac{1}{2}X_{\bar{p}}^T\mathbf{P}X_{\bar{p}} \tag{4}$$

where

$$\mathbf{P}=\begin{pmatrix} P & \alpha P_2 & \cdots & \alpha P_{\bar{p}} \\ \alpha P_2^T & P_2^T P_2 & \cdots & P_2^T P_{\bar{p}} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha P_{\bar{p}}^T & P_{\bar{p}}^T P_2 & \cdots & P_{\bar{p}}^T P_{\bar{p}} \end{pmatrix} \tag{5}$$

with $\alpha \in \mathbf{R}$, P is an SPD constant matrix of $\mathbf{R}^{n\times n}$ and P_j constant matrices of $\mathbf{R}^{n\times n^j}$. The matrices P, P_2, …, $P_{\bar{p}}$ are determined by the computation of a set of algebraic equations (6) for P, (7) for P_p with p even and (8) for P_p with p odd, respectively (Khayati and Benabdelkader, 2012):

$$PF_1+F_1^T P+H_1^T QH_1-PG_0R^{-1}G_0^T P=0 \tag{6}$$

$$\left(I_{n^{p+1}}+U_{n^p\times n}\right)\mathscr{F}_p^T\mathscr{T}_p^T\cdot vec\left(\tilde{P}_p\right)=\mathscr{H}_p \tag{7}$$

$$\tilde{\mathscr{F}}_p^T\cdot vec\left(\tilde{P}_p\right)=\tilde{\mathscr{H}}_p \tag{8}$$

where $\mathscr{F}_p=\left(F_1-G_0R^{-1}G_0^TP\right)\otimes I_{n^p}$, $\mathscr{H}_p=\sum_{\substack{i,j,b,c=1\\ i+j+k+b+c+d=p+3}}^{p-1}\sum_{k,d=0}^{p-1} vec\left(W_{ijk}^T R^{-1}W_{bcd}\right)-\sum_{\substack{i,j=1\\ i+j+k=p+2}}^{p-1}\sum_{k=1}^{p}\left[vec\left(V_{ij}^T F_k\right)+vec\left(F_k^T V_{ij}\right)\right]-\sum_{\substack{i,j=1\\ i+j=p+1}}^{p} vec\left(H_i^T QH_j\right)$, $\mathscr{T}_p=\left(T_p^+\otimes I_n\right)\mathscr{D}_{p+1}^{(n)}$ and $\tilde{\mathscr{F}}_p=\mathscr{T}_p\mathscr{F}_p\left(I_{n^{p+1}}+U_{n\times n^p}\right)T_{p+1}$ be rectangular full rank matrices of $\left(\left(n\cdot\tau_p^{(n)}\right)\times n^{p+1}\right)$ and $\left(\left(n\cdot\tau_p^{(n)}\right)\times\tau_{p+1}^{(n)}\right)$, respectively, and finally $\tilde{\mathscr{H}}_p=T_{p+1}^T\cdot\mathscr{H}_p$. Note $V_{ij}=mat_{n^{i+j-1}\times n}^T\left[vec\left(P_{i(j)}^T P_{j(i)}\mathscr{D}_j^{(n)}\right)\right]$ and $W_{ijk}=mat_{n^{i+j+k-1}\times m}^T\left[vec\left(V_{ij}^T G_k\right)\right]$ where $P_{1(1)}=P_1$, with P_1 is obtained from the Cholesky decomposition $P=P_1^TP_1$, $P_{1(j)}=\alpha I_n$

$\forall j=2,3,\ldots$ and $P_{i(j)}=P_i \;\; \forall i=2,3,\ldots \;\; \forall j=1,2,\ldots$. $\mathscr{D}_j^{(n)}$ is the square j-differential Kronecker matrix of $\mathbb{R}^{n^j}$ introduced in (Khayati and Benabdelkader, 2012). If P, P_2, …, $P_{\bar{p}-1}$ are known, then $\tilde{P}_p$ can be calculated as a solution of the linear equations (7) and (8), and thus, $P_p=\tilde{P}_p T_p^+$ is deduced. Note that $\left(F_1-G_0R^{-1}G_0^TP\right)$ is a Hurwitz matrix (Rotella, 1988), $\mathscr{D}_{p+1}^{(n)}$ is a singular matrix for all integers $p=2,\ldots,\bar{p}$ and $\left(I_{n^{p+1}}+U_{n^p\times n}\right)$ is regular for p even and singular for p odd. The practical suboptimal control $\bar{u}(x)$ introduced by (Khayati and Benabdelkader, 2012) is given by:

$$\bar{u}(x)=-\sum_{p=1}^{\bar{p}_g}K_p x^{|p|} \tag{9}$$

with $\bar{p}_g=2\bar{p}+g-1$ and $K_p=R^{-1}\sum_{\substack{i,j=1\\ i+j+k=p+1}}^{\bar{p}}\sum_{k=0}^{g}W_{ijk}$.

4. Stability of the Sub-optimal State-feedback

To investigate the stability of the closed loop system, consider $V(x)$ given by (4) as a Lyapunov candidate function. Note that $V(x)$ is radially unbounded continuous function and its derivative exists and is continuous. From (4) and (5), if

$$\mathbf{P}\left(\alpha,P,P_2,\ldots,P_{\bar{p}}\right)>0 \tag{10}$$

holds, then the Lyapunov candidate function $V(x)$ is positive definite; that is $V(x)>0$, $\forall x\neq 0$. The time derivative of the LF along the trajectories of the closed loop system (1) and (9) is given by

$$\dot{V}(x)=\frac{\partial V}{\partial x}^T\cdot\dot{x}(t)=\frac{\partial V}{\partial x}^T F(x)-\bar{u}(x)^T R\bar{u}(x) \tag{11}$$

Let us define B_1 and C_1 by $G_0R^{-1}G_0^T$ and $H_1^TQH_1$, respectively. We assume the triplet (F_1,B_1,C_1) is stabilizable-detectable. Note that if P a solution of an algebraic Ricatti equation (ARE) exists, then it is the unique SPD matrix solution of the optimal control on the linearized system and $\left(F_1-G_0R^{-1}G_0^TP\right)$ is a Hurwitz matrix. Thus, the linearized system is asymptotically stable (Rotella and Tunguy, 1988). Moreover, the nonlinear closed loop system (1) and (9) is locally asymptotically stable (LAS) and $\exists\, x\neq 0$ *s.t.* $\dfrac{\partial\left(x^TPx\right)}{\partial t}<0$. In the following, assume $\left\{x\in\mathbb{R}^n\setminus\{0\}\,|\,\dot{V}<0\right\}\neq\varnothing$. Consider the closed ball $\mathcal{B}(\delta)=\left\{x\in\mathbb{R}^n\,|\,\|x\|\leq\delta\right\}$. Assuming that $\exists\alpha$ *s.t.* (11) holds; *i.e.* $V(x)>0$, $\forall x\in\mathbb{R}^n$, $\mathcal{B}(\delta)$ is an estimate of the DA if $\mathcal{B}(\delta)\subset\Delta=\left\{x\in\mathbb{R}^n\,|\,\dot{V}<0\right\}\cup\{0\}$ (Chesi, 2009; Chesi, 2003). The computation of the maximum δ *s.t.* $\mathcal{B}(\delta)\subset\Delta$, *i.e.* (1) and (9) is LAS and the LEDA is given by $\mathcal{B}(\gamma)$ where (Chesi, 2009)

$$\gamma = \inf_{x \in \mathbb{R}^n \setminus \{0\} \ s.t. \ \dot{V}(x)=0} \|x\| \tag{12}$$

5. LEDA Computation of the Closed-loop System

Let $\mathbb{S}(\delta)$ be a sphere given by (Chesi, 2003): $\mathbb{S}(\delta) = \left\{ x \in \mathbb{R}^n \mid \|x\| = \delta \right\}$. The problem (12) turns out that

$$\gamma = \sup\left\{ \bar{\delta} \mid \dot{V}(x) < 0, \forall x \in \mathbb{S}(\delta), \forall \delta \in \left(0, \bar{\delta}\right] \right\} \tag{13}$$

The terms $\frac{\partial V}{\partial x}^T F(x)$ and $\bar{u}(x)^T R \bar{u}(x)$ are polynomials in x of degrees $2\bar{p} + f - 1$ and $2\bar{p}_g = 4\bar{p} + 2g - 2$, respectively. For any $\delta > 0$, $\forall x \in \mathbb{S}(\delta)$, we write

$$\dot{V}(x) = \sum_{k=1}^{2\bar{p}-1} \sum_{l=1}^{f} x^{|k|T} v_k^T F_l x^{|l|} - \sum_{i=1}^{\bar{p}_g} \sum_{j=1}^{\bar{p}_g} x^{|i|T} K_i^T R K_j x^{|j|} \tag{14}$$

with $v_k = \sum_{\substack{i=1 \\ i+j-1=k}}^{\bar{p}} \sum_{j=1}^{\bar{p}} V_{ij}$. Using the non-redundant VPS in $\tilde{x}^{|i|}$ and the vector notations of $X_{\bar{p}}$ and $\tilde{X}_{\bar{p}}$ introduced in Section 2, without loss of generality, we assume that $\exists \Gamma_d = \Gamma_d^T > 0$ *s.t.* $\sum_{i=1}^{\bar{p}_g} \sum_{j=1}^{\bar{p}_g} x^{|i|T} K_i^T R K_j x^{|j|} = \tilde{X}_{\bar{p}_d}^T \Gamma_d \tilde{X}_{\bar{p}_d} + w_e(x) + w_o(x)$. The term $w_o(x)$ (*resp.* $w_e(x)$) is polynomial in odd (*resp.* even) vector power terms of x of order $2\bar{p}_o + 1$ (*resp.* $2\bar{p}_e$). The integers $\bar{p}_d$, $\bar{p}_e$ and $\bar{p}_o$ are *s.t.* $0 \le \bar{p}_d \le \bar{p}_g$, $0 \le \bar{p}_e \le \bar{p}_g$ and $0 < p_o \le 2\bar{p}_g$. Thus, we obtain

$$\dot{V}(x) = -\sum_{i=1}^{\bar{p}_s} \tilde{x}^{|i|T} S_{ii}(\beta_{1i}) \tilde{x}^{|i|} - \frac{1}{2} \sum_{i=1}^{\bar{p}_t} \left[\tilde{x}^{|i|T} S_{i,i+1}(\beta_{2i}) \tilde{x}^{|i+1|} + \tilde{x}^{|i+1|T} S_{i,i+1}^T(\beta_{2i}) \tilde{x}^{|i|} \right] - \tilde{X}_{\bar{p}_d}^T \Gamma_d \tilde{X}_{\bar{p}_d} \tag{15}$$

$\forall i \in \{1, 2, \ldots, \bar{p}_s\}$, $S_{ii}(\beta_{1i}) \in \mathbb{R}^{\tau_i^{(n)} \times \tau_i^{(n)}}$ is the SMR matrix of terms in $-\sum_{k=1}^{2\bar{p}-1} \sum_{l=1}^{f} x^{|k|T} v_k^T F_l x^{|l|} + w_e(x)$ of order $2i$, and $\beta_{1i} \in \mathbb{R}^{\sigma(n.\tau_i^{(n)})}$ a free vector with $\sigma\left(n, \tau_i^{(n)}\right) = \frac{1}{2} \tau_i^{(n)} . \left(\tau_i^{(n)} + 1\right) - \tau_{2i}^{(n)}$ and $\tau_i^{(n)}$ stands for binomial coefficients (Mtar et *al.*, 2009). $S_{i,i+1}(\beta_{2i})$ is the RMR of terms in $-\sum_{k=1}^{2\bar{p}-1} \sum_{l=1}^{f} x^{|k|T} v_k^T F_l x^{|l|} + w_o(x)$ of order $2i+1$. $\bar{p}_s = \max\left(\bar{p} + \frac{f}{2} - \frac{1}{2}, \bar{p}_e \right)$ and $\bar{p}_t = \max\left(\bar{p} + \frac{f}{2} - \frac{3}{2}, \bar{p}_o \right)$ if f is odd, and $\bar{p}_s = \max\left(\bar{p} + \frac{f}{2} - 1, \bar{p}_e \right)$ and $\bar{p}_t = \max\left(\bar{p} + \frac{f}{2} - 1, \bar{p}_o \right)$ if f is even. Finally, we obtain

$$\dot{V}(x) = -\tilde{X}_{\bar{p}_m}^T \mathbf{S}(\beta)\tilde{X}_{\bar{p}_m} - \tilde{X}_{\bar{p}_d}^T \Gamma_d \tilde{X}_{\bar{p}_d} \tag{16}$$

where $\bar{p}_m = \max(\bar{p}_s, \bar{p}_t)$. $\mathbf{S}(\beta)$ is a general block-tridiagonal matrix in form $\mathbf{S}(\beta) = \begin{pmatrix} S_{11} & S_{12} & 0 & 0 & \dots & 0 \\ S_{12}^T & S_{22} & S_{23} & 0 & \dots & 0 \\ 0 & S_{23}^T & S_{33} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & & 0 \\ 0 & & & & S_{\bar{p}_{m-1},\bar{p}_{m-1}} & S_{\bar{p}_m,\bar{p}_{m+1}} \\ 0 & & & 0 & S_{\bar{p}_m,\bar{p}_{m+1}}^T & S_{\bar{p}_m,\bar{p}_m} \end{pmatrix}$. The decision variables β are set by the concatenation of all β_{1i} and β_{2i}.

5.1. Case of $\bar{p}_d = \bar{p}_m$

We have $\dot{V}(x) < 0$ if the LMI $\mathbf{S}(\beta) + \Gamma_d > 0$ holds in the free decision variable β. Thus if (10) holds and the LMI $\mathbf{S}(\beta) + \Gamma_d > 0$ is feasible in β, then the sub-optimal state-feedback (1) and (9) is GAS.

5.2. Case of $\bar{p}_d \neq \bar{p}_m$

Consider v the least common multiple of $\bar{p}_d$ and $\bar{p}_m$, *i.e.* $\exists (v_m, v_d) \in \mathbb{N}^2 | \{(0,0)\}$ *s.t.* $v = v_m \bar{p}_m = v_d \bar{p}_d$. $\forall x \in \mathbb{S}(\delta)$, we have $\left\|\chi_{\bar{p}_m}^{(v_m)}\right\|^2 = \sum_{i=0}^{v_m-1} \delta^{2i\bar{p}_m} = \delta_m^2$, $\left\|\chi_{\bar{p}_d}^{(v_d)}\right\|^2 = \sum_{i=0}^{v_d-1} \delta^{2i\bar{p}_d} = \delta_d^2$ and $\chi_{\bar{p}_m}^{(v_m)} \otimes X_{\bar{p}_m} = \chi_{\bar{p}_d}^{(v_d)} \otimes X_{\bar{p}_d} = X_v$. Thus, (16) is equivalent to $\dot{V}(x) = -X_v^T \left(\frac{1}{\delta_m^2} I_{\xi_m} \otimes \mathbf{S}^+(\beta) + \frac{1}{\delta_d^2} I_{\xi_m} \otimes \Gamma_d^+ \right) X_v$, with $\mathbf{S}^+(\beta) = \mathbf{T}_{\bar{p}_m}^{+T} \mathbf{S}(\beta) \mathbf{T}_{\bar{p}_m}^+$ and $\Gamma_d^+ = \mathbf{T}_{\bar{p}_d}^{+T} \Gamma_d \mathbf{T}_{\bar{p}_d}^+$. $\mathbf{T}_{\bar{p}_d}^+$ (*resp.* $\mathbf{T}_{\bar{p}_m}^+$) is obtained from $\mathbf{T}_{\bar{p}_d} \in \mathbb{R}^{N_{\bar{p}_d} \times \tau_{\bar{p}_d}}$ (*resp.* $\mathbf{T}_{\bar{p}_m} \in \mathbb{R}^{N_{\bar{p}_m} \times \tau_{\bar{p}_m}}$) using notations introduced in Section 2, $\xi_d = 1 + n^{\bar{p}_d} + n^{2\bar{p}_d} + \dots + n^{(v_d-1)\bar{p}_d}$, $\xi_m = 1 + n^{\bar{p}_m} + n^{2\bar{p}_m} + \dots + n^{(v_m-1)\bar{p}_m}$. Based on the assumption $\Gamma_d = \Gamma_d^T > 0$, $\Gamma_d^+ = \mathbf{T}_{\bar{p}_d}^{+T} \Gamma_d \mathbf{T}_{\bar{p}_d}^+$ is SPD. Let $\bar{\Theta}$ be its Cholesky's factor, *i.e.* $\Gamma_d^+ = \bar{\Theta}^T \bar{\Theta}$. Then, it follows $\forall x \in \mathbb{S}(\delta)$, $\dot{V}(x) < 0$ is equivalent to

$$\left(I_{\xi_d} \otimes \bar{\Theta}^{-T}\right)\left(I_{\xi_m} \otimes \mathbf{S}^+(\beta)\right)\left(I_{\xi_d} \otimes \bar{\Theta}^{-1}\right) + \bar{\delta} I_{\xi_d N_{\bar{p}_d}} > 0 \tag{17}$$

Note the factor $\bar{\delta} = \frac{\delta_m^2}{\delta_d^2}$ is a function of δ. Notice that if $v_m > v_d \Leftrightarrow \bar{p}_m < \bar{p}_d$ then $\bar{\delta} > 1$ and monotically increasing with δ. The following result holds.

a) Sub-case $\bar{p}_m < \bar{p}_d$: $\forall \beta$, $\exists \bar{\delta} > 1$ *s.t.* the LMI (18) holds. Thus, if (10) holds, then the sub-optimal state-feedback (1) and (9) is GAS.

b) Sub-case $\bar{p}_m > \bar{p}_d$: Given $\upsilon \in \mathbb{R}$, consider the LMI

$$\left(I_{\xi_d} \otimes \bar{\Theta}^{-T}\right)\left(I_{\xi_m} \otimes \mathbf{S}^{+}(\beta)\right)\left(I_{\xi_d} \otimes \bar{\Theta}^{-1}\right) - \upsilon I_{\xi_d N_{\bar{p}_d}} > 0 \tag{18}$$

In the vector β and the scalar υ. If $\exists \upsilon \geq 0$ *s.t.* the LMI (18) holds, then the LMI constraint holds $\forall \bar{\delta} > 0$, then we select $\bar{\delta} < 1$ and we have $\bar{\delta}$ decreasing *w.r.t* δ (*i.e.* $\delta \to \infty$ as $\bar{\delta} \to 0$). Thus, if (10) holds and the LMI (18) is feasible in $\upsilon \geq 0$ and β, then the sub-optimal closed-loop (1) and (9) is GAS.

c) Sub-case $\bar{p}_m > \bar{p}_d$ *and* $\exists \upsilon \in (-1, 0)$*:* then, a lower bound $\bar{\gamma}$ of γ, introduced in (14), is given by: $\hat{\gamma} = \arg_{\delta} \frac{\left(1+\delta^2+\ldots+\delta^{2(\nu_m-1)\bar{p}_m}\right)}{\left(1+\delta^2+\ldots+\delta^{2(\nu_d-1)\bar{p}_d}\right)} = (-\bar{\upsilon})$, where $\bar{\upsilon}$ is a solution of the following eigen-value problem (EVP): $\bar{\upsilon} = \max \upsilon$ subject to $-1 < \upsilon < 0$ and LMI (18). If $\arg\max_{\upsilon}$ of this EVP is negative, then the linear inequality constraint $-1 < \upsilon < 0$ corresponds to $\tilde{\delta} < 1$ as $\bar{p}_m > \bar{p}_d$.

6. Conclusion

In this paper, the analysis of the stability of the closed loop infinite horizon control is discussed in terms of the LMI feasibility problem. Then, the problem of computing the LEDA is cast as a convex EVP design. The contribution of this work is to develop a systematic LF based approach and a practical KP-based design for a large scale of nonlinear systems operating inside wider DA conditions.

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Curriculum Vitae

I was born on September 8, 1977 in Bizerte, Tunisia. I grew up in Tunis, Tunisia and guraduated from the ENIT (école nationale des ingénieurs de Tunis) in June 2001 with BS in Mechanical Engineering. I worked as a quality engineer from 2001 to 2008 in many multinational companies designing and manufacturing components and subsystems for the automotive industry. From 2009 to 2012 I worked at the RMC instrumentation lab as a research assistant. From 2013 and up to date I am working as a quality engineer at WTG in Quebec, Canada, a company designing and manufacturing sealing systems for the automotive industry.